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International capital flows, portfolio composition, and the stability of external imbalances☆

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This paper develops a simple and tractable model of net capital flows in which time-varying gross country portfolios are an essential element in current account imbalances. The main constituents of country portfolios in the model are general derivatives, which could be interpreted as nominal bond assets and liabilities in particular. Under very weak conditions, the world wealth distribution is stationary. Stationarity is generated by movements in derivative (i.e., bond) risk-premia such that the return on a debtor country's gross liabilities is less than the return on its gross assets. This is well known feature of the US international investment position. We also provide suggestive evidence that a similar property holds more widely for a sample of advanced and emerging market countries.

1. Introduction

The last three decades have seen an unprecedented increase in two-way financial flows between countries. Even after the Great Financial Crisis, there has been a continued increase in the size of gross external assets and liabilities, and at the same

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time a continued presence of large current account imbalances across countries (Lane and Milesi-Ferretti, 2007b; Lane and Milesi-Ferretti, 2018).

In an environment with large holdings of gross assets and liabilities denominated in different currencies, asset classes, or maturity structures, the interpretation of the current account based on a simple measurement of net-foreign assets (NFA, hereafter) may be quite misleading (see Obstfeld (2012)). Borio (2016) argues that from a policy perspective, the central role of the current account in the G20 policy debate is not informative for understanding external imbalances and instead it is more important to focus attention on the structure of external assets and liabilities. In fact, financial globalization may facilitate larger current account positions than would be consistent with capital markets based on one-way capital flows (Obstfeld, 2012).[1]

This paper develops a simple, analytically tractable general equilibrium model of portfolio choice in a two-country one-good world economy with incomplete markets and trade in derivatives, which may also be interpreted as nominal bonds. The model provides insights into the relationship between gross external asset and liability positions and the determination of the current account. The analysis shows that endogenous portfolio composition, involving movements in gross positions, is essential in facilitating international net capital flows between countries. Movements in net foreign assets are generated by gross assets and gross liabilities moving in the same direction, giving rise to time-varying portfolios and asset returns. The dynamics of *gross* positions in assets with different risk characteristics are essential ingredients in facilitating *net* capital flows.

An important building block of the paper is that endogenous variation in gross portfolio positions and real returns ensure a stationary world wealth distribution, implying that countries' external imbalances are self-correcting. The key mechanism ensuring stationarity and self-correction is that asset returns move so as to reduce the cost of borrowing for debtor countries. Countries with negative net foreign asset positions tend to have higher excess returns on their assets relative to their liabilities and vice versa.

The key feature of the model is the ability of derivatives to share country-specific risk in an incomplete markets environment. Trade in derivative assets allows for effective portfolio diversification. We start with a stochastic, continuous time framework with country-specific technology shocks. If financial markets consisted only of a real risk-free bond, as in the textbook one-good current account model, because productivity shocks are permanent, then there would be no gains from trade between countries at all. But trade in derivatives allows for countries to share risk by holding a diversified portfolio of domestic and foreign derivatives. Because risk sharing is limited, country specific shocks cause movements in relative national wealth levels across countries. This causes time-variation in derivative returns and portfolio shares. Movements in portfolio holdings, or gross positions, are in turn associated with net capital flows between countries. Thus, current account movements are inherently tied to the adjustment of national portfolios and two-way capital flows. For instance, a country experiencing net capital inflows may be simultaneously issuing home derivatives, but purchasing foreign derivatives.[2] In our model, derivatives are represented as zero net-supply assets, and defined simply by the fact that their real return covaries with national technology shocks. So long as shocks are not perfectly correlated across countries, derivatives allow for international risk-sharing.

We derive a novel condition for stationarity in the world distribution of wealth in the presence of derivative trade. So long as derivatives allow some cross country risk-sharing, the world wealth distribution is stationary. Moreover, there is a simple and highly intuitive explanation of the stationarity result: asset returns tend to move to the disadvantage of creditor countries and to the advantage of debtor countries. This ensures that as a country's relative wealth position deteriorates, its cost of borrowing also falls, encouraging it to invest in its domestic technology, and increase its expected growth rate. More specifically, we find that debtor countries face a lower return on their gross liabilities than they receive on their gross assets, while creditor countries face the opposite situation. In this way stationarity in the wealth distribution is tied directly to time-variation in asset returns and portfolio composition.

We show the results first in a baseline version of the model in which there is trade in derivatives and a real risk-free bond. As a special case, we also consider trade in nominal bonds, whose payoffs are denominated in national currency, and are subject to national monetary policy shocks and exchange rate shocks. The nominal bond trading economy provides a laboratory for taking the model to the data. The feature that overall international investment returns are negatively related to net foreign asset positions seems consistent with observations. It is widely acknowledged that the US, as the world's largest debtor, receives a higher return on its gross external assets than it pays on its gross external liabilities (Gourinchas and Rey, 2014). In Section 3.2.2 below we provide illustrative evidence of a similar property for a large sample of countries.

Furthermore, we show that the stationarity results apply in a number of different extensions of the model, restricting trade to nominal bonds only, to trade in just one country's nominal bond, or in environment extended to allow for trade in both nominal bonds and equity. We find that nominal bonds act as a complement to trade in a real risk-free bond. By contrast, nominal bonds represent a substitute for trade in equity. In the baseline model, we assume no direct trade in claims to the economy's production technology (equity). Unrestricted equity trade would imply complete markets. In an extension however, we allow limited equity trade. But even then, agents may hold only a small share of foreign equity. The reason is that the risk-sharing through nominal bonds may remove the need for trade in equity. Thus, the presence of nominal bond trade may imply a home bias in equity holdings.[3]

Our paper is related to a number of strands in the literature on risk-sharing and international financial markets. An important element of our model with nominal bonds is that in an environment of incomplete markets, the return-distribution of nominal

[1] Of course, financial linkages may also carry substantial risks in the presence of financial frictions, maturity and currency mismatches, as evidenced in (Pavlova and Rigobon, 2008; Perri and Quadrini, 2018; Devereux and Yu, 2020).

[2] Forbes and Warnock (2012) empirically show that two-way capital flows, particularly extreme capital flows, are significantly associated with global factors, especially global risk.

[3] See, for instance, Engel and Matsumoto (2009), Coeurdacier et al. (2010), Heathcote and Perri (2013), Coeurdacier and Gourinchas (2016) for more discussions on local currency bonds that are used to hedge real exchange rate risk and income risk.

assets plays a role in cross country risk-sharing. This insight was noted in early papers by Svensson (1989) and Bacchetta and van Wincoop (2000). A similar mechanism underlies the model of Devereux and Sutherland (2008). At a more general level, other recent papers have noted the risk sharing properties of non-contingent bonds in two-good frameworks, where bond returns in different currencies or goods are affected by endogenous movements in the terms of trade. In particular, this property holds in the models of Engel and Matsumoto (2009), Heathcote and Perri (2013) and Kollmann (2006), who show how endogenous relative prices may support complete markets even with home bias in equities. Also, Coeurdacier et al. (2010) and Coeurdacier and Gourinchas (2016) explore the role of bonds in hedging terms of trade and real exchange rate risks. The latter paper provides strong empirical evidence for the role of bond positions in accounting for home bias in G7 countries. Coeurdacier and Rey (2013) survey the recent literature on home bias in equity portfolios.

More generally, Devereux and Sutherland (2010, 2011), Tille and van Wincoop (2010) and Hnatkovska (2010) independently develop alternative local approximation methods to solve country portfolios in dynamic stochastic general equilibrium models with incomplete markets. But their local approximation does not allow them to explore the issue of stationarity. Several other authors investigate the global dynamics of country portfolios. Pavlova and Rigobon (2008) construct a continuous-time stochastic model of portfolio choices with portfolio constraints, and focus on aspects of asset pricing and the international transmission of stock prices. Brunnermeier and Sannikov (2015) develop a continuous-time stochastic two-country two-good model with incomplete markets à la Cole and Obstfeld (1991), and explore pecuniary externalities due to excessive short-term credit flows. Devereux-Yu-contagion-RES numerically explore country portfolio dynamics and financial contagion in a two-country environment with occasionally binding credit constraints. Our paper is complementary to the studies above by focusing on the stationarity of the country wealth distribution and the stability of external imbalances.[4]

In addition, the source of stationarity in the wealth distribution here differs from that of previous literature. In a version of the neoclassical growth model with idiosyncratic endowment risk, Aiyagari (1994) shows that precautionary savings can support a stationary wealth distribution, as long as agents are not too patient (see also Krusell and Smith, 1998; Carroll, 2011). With precautionary saving, stationarity is ensured by poor agents saving more and wealthy agents saving less, while all saving is done in the form of an aggregate risk-free asset.[5] Schmitt-Grohe and Uribe (2003) quantitatively compare a set of alternative mechanisms for ensuring stationarity in locally approximated small open economy models. A model with a debt-elastic interest-rate premium in these models might be most related. Nevertheless, in our model stationarity is associated with aggregate shocks, which change the composition of real returns. But the presence of derivatives (or nominal bonds) is also critical. In order to ensure a stationary wealth distribution, agents must continually adjust not only their aggregate savings, but also the portfolio composition of their savings.[6]

Our paper falls in a second strand of literature on external imbalances and current account adjustment. Financial globalization leads to a wide distribution of current accounts and net external investment positions (Lane and Milesi-Ferretti, 2007b; Lane and Milesi-Ferretti, 2018).[7] Net external deficits can be adjusted either through future trade surpluses (trade channel) and/or excess returns of external assets over liabilities (valuation channel). Gourinchas and Rey (2007b) and the follow-up researchers illustrate that the United States earned a higher rate of return on external assets than it paid on liabilities over various data samples as the US net foreign asset position turned negative, and the valuation channel contributed to a large fraction of its cyclical external financial adjustment.[8] Our model implicitly includes both the direct returns and the valuation channel, providing a theoretical justification for this characteristic of external financial adjustment. Section 3.2.2 explores this property using data on observed international investment positions for a wide group of countries. Based on a sample of 19 advanced countries and 32 emerging and developing countries during 1980–2016, we find evidence that the ratio of a country's net foreign assets to GDP has a significantly negative effect on the excess return of external assets over liabilities. The empirical evidence is therefore consistent with our model, which predicts a negative relationship between country's net foreign assets and the excess return on domestic investment in the rest of the world. This self-correcting channel acts so as to stabilize the wealth distribution across countries in the long run.

The rest of the paper is structured as follows. The next section develops the basic model with general derivative trading and derives the main results, providing a full characterization of international portfolio holdings as well as establishing the stability properties of net foreign assets. Section 3 explores nominal bond trading as a special case of Section 2, but considers national monetary policy shocks and exchange rate shocks. A subsection reports some illustrative empirical evidence for the relationship between net foreign assets and excess returns on the external portfolio. Section 4 extends the model to a number of different environments. Section 5 presents some final remarks.

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[4] In an incomplete markets environment, Heaton and Lucas (1996) and Krusell and Smith (1998) develop numerical methods for analyzing asset pricing and risk sharing. Kubler and Schmedders (2003, 2005) prove the existence of a stationary equilibrium in asset pricing models with incomplete markets and collateral constraints, and propose a numerical algorithm to obtain optimal policy rules including portfolio decisions. In essence, our model is a multi agent version of Merton (1971) with restrictions on asset trade. This makes the model amenable to a large number of applications, although it does restrict its applicability in explaining some puzzles, such as those related to asset pricing.

[5] Perri and Quadrini (2018) and Devereux and Yu (2020) make use of precautionary savings to obtain a stationary distribution of wealth between investors and savers.

[6] An alternative mechanism for ensuring a stationary wealth distribution is through endogenous movements in the terms of trade. See Cole and Obstfeld (1991), Acemoglu and Ventura (2002) and Brunnermeier and Sannikov (2015).

[7] There are many reasons generating such external imbalances. For instance, Caballero et al. (2008) explore the role of countries' heterogeneous ability to produce high quality assets, while Mendoza et al. (2009) focus on the heterogeneity of asset demands across borders. The imbalances in our model are simply driven by exogenous technology shocks.

[8] There are three waves of literature on estimating the excess returns of external liabilities over assets for the US. More recent reviews can be found in Curcuru et al. (2013) and Gourinchas and Rey (2014).

2. The model: Trade in derivatives

2.1. A basic setup

We take a one-good two-country model of a world economy. In each country there is a risky linear technology which uses capital and generates expected instantaneous return α_i with standard deviation σ_i, where $i = h$ or f, signifying the 'home' or 'foreign' country. Capital can be turned into consumption without any cost. The return on technology i (in terms of the homogeneous good) is given by:

$$\frac{dQ_i}{Q_i} = \alpha_i dt + \sigma_i dB_i, \tag{1}$$

for $i = h$ or f, where dB_i is the increment to a standard Wiener process. That is, a shock represented by dB_i has permanent effects on wealth. For simplicity, we assume that the returns on the two technologies are independent, so that $\lim_{\Delta t \to 0} \frac{Cov_t(\Delta B_h(t+\Delta t), \Delta B_f(t+\Delta t))}{\Delta t} = 0$.

To study the dynamics of the world wealth distribution, we assume that financial markets are incomplete. Residents of one country cannot directly purchase shares in the technology of the other country (we partially relax this assumption in the analysis below). Risk-free bonds can be traded between the countries however. In addition, two zero-net-supply derivatives are introduced as partial risk sharing instruments for domestic technologies. That is, each derivative is issued to one country by the other country. In Section 3, these derivatives are interpreted as nominal bonds denominated in different currencies.

The payoff on derivative j, where $j = h$ or f, follows

$$R_j dt + \Delta_j dD_j, \tag{2}$$

where R_j is an endogenous deterministic real return on derivative j, and D_j is the increment to a standard Wiener process. Innovations ΔD_h and ΔD_f are assumed to be uncorrelated. Standard deviations Δ_j, $j = h, f$, are exogenously given.

In order to emphasize the key mechanisms at play, we present here a simplified, symmetric version of the model in which each derivative covaries with only one of the underlying country technologies.

Innovation ΔB_h is correlated with innovations of derivatives h and f as

$$\lim_{\Delta t \to 0} \frac{Cov_t(\Delta B_h(t+\Delta t), \Delta D_h(t+\Delta t))}{\Delta t} = \lambda_h^h, \tag{3}$$

and

$$\lim_{\Delta t \to 0} \frac{Cov_t\left(\Delta B_h(t+\Delta t), \Delta D_f(t+\Delta t)\right)}{\Delta t} = 0. \tag{4}$$

Thus, the home technology is correlated with derivative h, but not with derivative f. We allow for λ_h^h to be of either sign, but we assume that $\lambda_h^h \neq 0$, and $|\lambda_h^h| < 1$ In analogous fashion, we define λ_f^f as the limiting covariance between derivative f and the foreign technology, and assume that derivative f is uncorrelated with the home technology. Again, λ_f^f may be of either sign, but we assume that $\lambda_f^f \neq 0$ and $|\lambda_f^f| < 1$.

Given the above setup, the real risk-free instantaneous return r, as well as the deterministic returns on the two derivatives R_h and R_f are determined by world market equilibrium. Agents in each economy divide their wealth W_i, $i = h, f$ across holdings of the domestic technology, the real risk-free bond and the two separate derivatives. C_h denotes consumption of the representative home household. As shown in Section 4.1, most results presented below survive even if the real risk-free bond market is absent.

The budget constraint for the home country may then be written as:

$$dW_h = W_h\left[\omega_T^h(\alpha_h - r) + \omega_h^h(R_h - r) + \omega_f^h\left(R_f - r\right) + r\right]dt - C_h dt + W_h\left[\omega_T^h \sigma_h dB_h + \omega_h^h \Delta_h dD_h + \omega_f^h \Delta_f dD_f\right], \tag{5}$$

where ω_T^h, ω_h^h, and ω_f^h are the portfolio shares, respectively, of the domestic technology, and the two derivatives. Hence, $1 - \omega_T^h - \omega_h^h - \omega_f^h$ represents the share of the real risk-free bond.

Each country is populated by a continuum of identical agents. Preferences are identical across countries, and given by:

$$E_0 \int_0^\infty \exp(-\rho t) \ \ln C_i(t) dt, \tag{6}$$

where ρ is the rate of time preference. With preferences given by Eq. (6), the relevant measure of expected consumption growth in any equilibrium is the *risk-adjusted growth rate*, given by:

$$\lim_{\Delta t\to 0} E_t\left[\frac{\Delta\ \ln C_i(t+\Delta t)}{\Delta t}\right] = \lim_{\Delta t\to 0}\frac{E_t\left(\frac{\Delta C_i(t+\Delta t)}{C_i(t)}\right)-\frac{1}{2}Var_t\left(\frac{\Delta C_i(t+\Delta t)}{C_i(t)}\right)}{\Delta t}.$$

At any moment in time, an equilibrium in the market for the two derivatives determines the deterministic rates of return R_h and R_f. Derivative market clearing conditions are given as:

$$\omega_h^h W_h + \omega_h^f W_f = 0, \tag{7}$$

and

$$\omega_f^h W_h + \omega_f^f W_f = 0. \tag{8}$$

The above equations just say that the sum of derivative demands must add up to the world zero-net-supply. For example, $\omega_h^h < 0 < \omega_h^f$ implies that derivative h is issued to the foreign country by the home country.

As we show below, derivative trade will endogenously generate gains from trade in the real bond. Therefore, we take account of the market clearing condition in the real bond as:

$$\left(\omega_T^h + \omega_h^h + \omega_f^h - 1\right)W_h + \left(\omega_T^f + \omega_h^f + \omega_f^f - 1\right)W_f = 0. \tag{9}$$

2.2. Derivative trading equilibrium in a symmetric case

2.2.1. Optimal consumption and portfolio rules

To highlight the role of derivatives in fostering intertemporal trade, we will focus on the case where countries have symmetric drift and diffusion parameters, so that, $\alpha_h = \alpha_f = \alpha$, $\sigma_h = \sigma_f = \sigma$, $\Delta_h = \Delta_f = \Delta$, and $\lambda_h^h = \lambda_f^f = \lambda$.

With logarithmic utility, home country consumers follow the myopic consumption rule:

$$C = \rho W.$$

The optimal portfolio rules can be obtained as the solution to:

$$\begin{bmatrix} \omega_T^h \\ \omega_h^h \\ \omega_f^h \end{bmatrix} = \begin{bmatrix} \sigma^2 & \sigma\lambda\Delta & 0 \\ \sigma\lambda\Delta & \Delta^2 & 0 \\ 0 & 0 & \Delta^2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha - r \\ R_h - r \\ R_f - r \end{bmatrix} \tag{10}$$

A similar set of conditions hold for the foreign country.

Using Eq. (10) and the equivalent for the foreign country, the market clearing conditions of Eqs. (7), (8), and (9) may be solved for R_h, R_f, and r. Define $\theta = \frac{W_f}{W_h + W_f}$ as the ratio of foreign wealth to world wealth. The solution has a recursive structure. Given the consumption rule, equilibrium returns and portfolio holdings depend on θ. We may then write the solution for nominal interest rates and the world risk-free rate as $R_h(\theta)$, $R_f(\theta)$, and $r(\theta)$. When markets are incomplete, θ will be time-varying, and therefore so are rates of return and portfolio shares. The dynamics of θ may be constructed from the wealth dynamics (5) and the equivalent process for the foreign country.

2.2.2. Alternative financial market configurations

To provide a reference point, we first characterize an equilibrium under two extreme asset market structures, where only real risk-free bonds are traded, and when there is free trade in claims on each country's technology. Then we analyze the equilibrium with derivative trade.

With only trade in a real risk-free bond, and with the fact that shocks in the model are permanent, there are no gains from trade between countries at all. Real risk-free bonds will not be traded, since the autarky risk-free rate on the real bond in each country is identical, given by $r^A = \alpha - \sigma^2$. In this case, $\omega_T^i = 1$. Each country's wealth is equal to its physical capital stock. A permanent home technology shock dB_h will change home consumption and wealth in the same proportion, leaving the real risk-free rate unchanged. The shock will have no affect at all on the foreign economy. In this case, the risk-adjusted consumption growth rate is $\alpha - \rho - \frac{1}{2}\sigma^2$.

If shares in each country's technology were freely tradable across countries, financial markets would be effectively complete. Trade in real risk-free bonds or derivatives would then be redundant. The equilibrium share of each technology (home and foreign) will be one half, and the equilibrium risk-free rate on the real bond will be $r^C = \alpha - \frac{1}{2}\sigma^2$. Risk-pooling under complete markets implies a higher risk-free interest rate than in autarky. In this case, all technology shocks are equally shared among home and foreign consumption. The risk-adjusted consumption growth rate is then $\alpha - \rho - \frac{1}{4}\sigma^2$.

2.2.3. Equilibrium behavior in asset pricing and optimal portfolios

Now allow for trade in derivatives. Since the real returns on the two derivatives are risky, and co-vary in different ways with the home and foreign technologies, the two countries will have different demands for derivatives. For instance, in the home economy under autarky, the equilibrium expected real interest rate on derivative h is $R_h^A = r^A + \sigma\lambda\Delta$ with $r^A = \alpha - \sigma^2$. This includes a risk premium term $\sigma\lambda\Delta$. When $\lambda > 0$, derivative h is a bad hedge against technology risk, and must have a return higher than the autarky risk-free rate r^A. The home country autarky equilibrium interest rate on derivative f is $R_f^A = r^A$. When $\lambda > 0$, the derivative f is a better hedge against consumption risk, and therefore carries a lower autarky return than the derivative h. The autarky returns on derivatives in the foreign country are just a mirror image of that in the home country. When $\lambda < 0$ of course, the opposite reasoning applies. But again, the autarky return on derivatives will differ across countries. This implies that there are gains from trade in derivatives.

For the equilibrium returns at the point of equal national wealth levels ($\theta = 0.5$),

$$\bar{r} = \alpha - \sigma^2 + \frac{1}{2}\sigma^2\lambda^2 \tag{11}$$

$$\overline{R}_h = \overline{R}_f = \overline{R} = \bar{r} + \frac{1}{2}\sigma\Delta\lambda \tag{12}$$

In the case of differences in wealth shares, i.e. $0 < \theta < 1$, the solutions for returns are

$$R_h(\theta) = \overline{R} - \frac{\left(1-\lambda^2\right)}{2\Gamma(\theta)}\sigma(2\theta-1)\lambda[(2\theta-1)\sigma\lambda + \Delta], \tag{13}$$

$$R_f(\theta) = \overline{R} + \frac{\left(1-\lambda^2\right)}{2\Gamma(\theta)}\sigma(2\theta-1)\lambda[-(2\theta-1)\sigma\lambda + \Delta], \tag{14}$$

$$r(\theta) = \bar{r} - \frac{1-\lambda^2}{2\Gamma(\theta)}\sigma^2(2\theta-1)^2\lambda^2, \tag{15}$$

where $\Gamma(\theta) = 1 - \lambda^2(1 - 2\theta(1-\theta)) > 0$.

Hence, equilibrium returns on derivatives and the risk-free bond are time varying, since with incomplete markets the wealth share θ will vary in response to the underlying technology shocks from each country. We see from Eqs. (11) and (15), the risk-free rate $r(\theta)$ is maximized at $\bar{r}$ ($=r(0.5)$) when $\theta = 0.5$, and it is minimized at the autarky rate $r^A = \alpha - \sigma^2$, when θ is 0 or 1. Since $\sigma^2\left(1-\frac{1}{2}\lambda^2\right) > 0$, $\bar{r}$ is always less than α, which means that despite varying wealth shares and net foreign asset positions neither country will ever take a short position in its own technology. In addition, the risk-free will always fall short of the complete markets rate $r^C = \alpha - \frac{\sigma^2}{2}$ for all values of θ.

Using Eqs. (10) and (13)–(15), and the equivalent for the foreign country, we may derive the equilibrium portfolio holdings under derivative trade. At the point of equal national wealth levels ($\theta = 0.5$),

$$\overline{\omega}_T^h = 1, \tag{16}$$

$$\overline{\omega}_h^h = -\overline{\omega}_f^h = -\frac{\sigma\lambda}{2\Delta}. \tag{17}$$

Thus, the net positions in both derivatives ($\overline{\omega}_h^h + \overline{\omega}_f^h$) and real bonds ($1 - \overline{\omega}_T^h - \overline{\omega}_h^h - \overline{\omega}_f^h$) are zero at $\theta = 0.5$. But the home (foreign) country will have a short position in derivative h (f) when $\lambda > 0$.

In the more general case where $0 < \theta < 1$, we have portfolio shares given as follows:

$$\omega_T^h(\theta) = 1 + \frac{1}{\Gamma(\theta)}\theta(2\theta-1)\lambda^2, \tag{18}$$

$$\omega_h^h(\theta) = \overline{\omega}_h^h - \frac{(2\theta-1)}{2\Delta\Gamma(\theta)}\sigma\lambda\left(1-\lambda^2(1-2\theta)\right), \tag{19}$$

$$\omega_f^h(\theta) = \overline{\omega}_f^h + \frac{(2\theta-1)}{2\Delta\Gamma(\theta)}\lambda\sigma\left(1-\lambda^2\right). \tag{20}$$

Given Eqs. (18)–(20), the net positions in derivatives and real bonds are obtained as:

$$\omega_h^h(\theta)+\omega_f^h(\theta)=-\frac{\theta(2\theta-1)\sigma\lambda^3}{\Delta\Gamma(\theta)}, \tag{21}$$

$$1-\omega_T^h(\theta)-\omega_h^h(\theta)-\omega_f^h(\theta)=-\frac{\theta(2\theta-1)\lambda^2(\Delta-\sigma\lambda)}{\Delta\Gamma(\theta)} \tag{22}$$

Then, the overall net foreign asset (NFA) position of the home country, relative to its wealth, will be:

$$1-\omega_T^h(\theta)=-\frac{\theta(2\theta-1)\lambda^2}{\Delta\Gamma(\theta)} \tag{23}$$

It is clear that these portfolio shares will be time-varying in response to changes in θ, even though returns on the real technologies and their correlation with derivative returns are symmetric across countries. From (23), the NFA position of home country is equal to zero when $\theta=0.5$. But as θ rises above 0.5, the home country NFA becomes negative. The country goes into debt in order to increase investment in the home technology. By contrast, when $\theta<0.5$, the home country is a net creditor.

2.3. Characteristics of the model with trade in derivatives

Here we discuss in detail the features of asset returns, portfolio dynamics, and asset trade in the two country model with trade in derivatives. First, we can summarize the main features of the model with the following proposition:
Using (13)–(15) and (18)–(23) we state the following propositions:

Proposition 1. *In the equilibrium with trade in derivative assets, assuming* $\lambda\neq0$ *and* $|\lambda|<1$.

a) The real risk-free rate lies between the autarky rate ($r^A=\alpha-\sigma^2$), *and the complete markets rate* ($r^C=\alpha-\frac{1}{2}\sigma^2$).

b) For $\lambda>0$ ($\lambda<0$), the home country holds a short (long) position in derivative h assets, and a long (short) position in derivative f assets for all values of θ.

c) When $\theta=0.5$, *the home country has a zero net position in derivative assets. For* $\lambda>0$, (<0), *the home country holds a negative (positive) net position in derivative assets for* $\theta>0.5$, *and conversely for* $\theta<0.5$.

d) For $\theta=0.5$, *the home country has a zero position in risk-free bonds. For* $\theta>0.5$, ($\theta<0.5$), *the home country has a negative (positive) position in risk-free bonds when* $\Delta>\lambda\sigma$. *The opposite applies when* $\Delta<\lambda\sigma$.

e) The home country has a negative (positive) net foreign asset (NFA) position for $\theta>0.5$ ($\theta<0.5$).

f) Let $\varrho(\theta)=R_h(\theta)-R_f(\theta)$ *be defined as the risk-premium on derivative h assets relative to derivative f assets. Then when* $\lambda>0$ ($\lambda<0$), $\varrho(\theta)$ *is negative (positive) for* $\theta>0.5$. *The opposite holds for* $\theta<0.5$.

Proof. a) From (15) we can establish that

$$r-r^A=\frac{\theta(1-\theta)\left(2-\lambda^2\right)\lambda^2\sigma^2}{\Gamma(\theta)}>0,$$

$$r-r^C=-\frac{\left(1-\lambda^2\right)\sigma^2\left(1-\lambda^2 2\theta(1-\theta)\right)}{2\Gamma(\theta)}<0,$$

b) This follows directly from Eqs. (19) and (20).

c) Follows directly from Eq. (21).

d) Follows directly from (22).

e) Follows directly from (23).

f) From eqs. (13) and (14), we have (for $\lambda > 0$) $\rho(\theta) = -\frac{\Delta\lambda\sigma(1-\lambda^2)(2\theta-1)}{\Gamma(\theta)} < 0$ (>0) as $\theta > 0.5$ ($\theta < 0.5$). The condition is reversed for $\lambda < 0$.

2.3.1. Portfolio diversification

It is clear from the proposition that an equilibrium with derivative asset trade allows for resource transfers across countries. The reason is that derivative assets have different characteristics with respect to hedging consumption risk for home and foreign consumers. When $\lambda > 0$ derivative h tends to have a high real return when returns on the home technology are high, and it thus represents a relatively bad hedge against home consumption risk. But derivative f represents a relatively good hedge against home consumption risk. On the other hand, for the foreign household, in the case $\lambda > 0$, derivative h is a good hedge and derivative f a bad hedge. In an equilibrium with derivative bond trade, home households will thus sell derivative h in return for derivative f, leading to the portfolio position described in Eqs. (19) and (20). This portfolio position allows the home country to receive a relatively high portfolio return when there is a positive shock to the foreign technology, and vice versa.

First focus on the point $\theta = 0.5$. Proposition 2 below shows that this is the modal point of θ. At this point, each country has a zero net external asset position. But the gross external assets will comprise a positive position in one asset, balanced by a negative position in the other asset. Thus, for the home country, in the case of $\theta = 0.5$, we have $\omega_h^h = \frac{\sigma\lambda}{2\Delta}$ and $\omega_f^h = -\frac{\sigma\lambda}{2\Delta}$. The absolute positions are higher, the greater the volatility of the productivity shock, and lower, the greater is the intrinsic volatility of the derivative asset.

The risk-sharing from portfolio diversification in derivatives reduces the volatility of consumption, and increases welfare. This is reflected in a higher real risk-free interest rate. At the point $\theta = 0.5$, the risk-free interest rate is $\bar{r} = \alpha - \sigma^2 + \frac{1}{2}\lambda^2\sigma^2$. This is closer to the complete markets value, the closer is λ to unity in absolute value, since the higher is $|\lambda|$, the better are derivatives as a hedge against consumption risk due to productivity shocks. The risk-adjusted consumption growth rate at $\theta = 0.5$ is written as $\alpha - \rho - \frac{1}{2}\sigma^2\left(1 - \frac{\lambda^2}{2}\right)$. When $\lambda = 0$, this is identical to that under autarky, while as $|\lambda| \to 1$, trading in derivative bonds alone attains the complete markets growth rate.

2.3.2. Capital flows

Net capital flows (or intertemporal trade) occur when the changes in a country's gross bond and derivative holdings do not sum to zero. The sum of wealth in the two countries is equal to the world capital stock, since capital is the only outside asset in the world economy. If portfolio diversification could sustain the complete markets allocation, then there would be no change in relative wealth across the two countries, and each country would maintain a constant share of the world capital stock. But because derivative trade cannot achieve the complete markets equilibrium, productivity shocks in one country will have a larger impact on that country's wealth than on the wealth of the other country. These changes in relative wealth levels give rise to net capital flows across countries.

Differentiating Eqs. (18)–(20) at $\theta = 0.5$, we see that a rise in θ has the following effect on the home country's portfolio:

$$\left.\frac{d\omega_h^h}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda\sigma}{\Delta(2-\lambda^2)}, \tag{24}$$

and

$$\left.\frac{d\omega_f^h}{d\theta}\right|_{\theta=0.5} = \frac{2\lambda\sigma\left(1-\lambda^2\right)}{\Delta(2-\lambda^2)}. \tag{25}$$

When $\lambda > 0$, the first expression is negative, and the second is positive. Hence, beginning at $\theta = 0.5$, a rise in foreign relative wealth will be followed by a rise in home gross borrowing in derivative h bonds, and a rise in gross lending in derivative f bonds. Such gross borrowing dominates gross lending, because we have

$$\left.\frac{d\left(\omega_h^h + \omega_f^h\right)}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda^3\sigma}{\Delta(2-\lambda^2)}. \tag{26}$$

Consequently, net derivative positions become negative when $\lambda > 0$.

In addition, there is a change in the holdings of *real* bonds. From Eq. (22) we have:

$$\left.\frac{d\left(1-\omega_h^h-\omega_f^h-\omega_T^h\right)}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda^2(\Delta-\lambda\sigma)}{\Delta(2-\lambda^2)}. \tag{27}$$

This may be positive or negative, depending on the sign of $\Delta - \lambda\sigma$.

Note also that the sum of Eqs. (26)–(27) is less than zero;

$$\left.\frac{d\left(1-\omega_T^h\right)}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda^2}{(2-\lambda^2)}.$$

That is, while at $\theta = 0.5$, the home country has a diversified portfolio but a zero net external balance, as the foreign country becomes larger in terms of world wealth, the home country becomes a recipient of foreign capital inflows.

With trade in real bonds alone, there are no international capital flows at all. How does the presence of derivative assets generate capital flows? The key feature is the interaction between changes in derivatives returns and *gross* bond holdings.

From the solutions for R_h and R_f, we find that:

$$\left.\frac{dR_h}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda\sigma\Delta\left(1-\lambda^2\right)}{2-\lambda^2},$$

and

$$\left.\frac{dR_f}{d\theta}\right|_{\theta=0.5} = \frac{2\lambda\sigma\Delta\left(1-\lambda^2\right)}{2-\lambda^2}.$$

The first expression is negative, while the second is positive, for $\lambda > 0$. Thus, a rise in the share of world wealth for the foreign country drives down the return on derivative h, while pushing up the return on derivative f bonds. Intuitively, as the foreign country increases its wealth, its portfolio preferences dominate the global bond markets. It increases its demand for derivative h, while increasing its supply of derivative f bonds.[9] This is reflected in the movements in the returns on derivative bonds.

The gross portfolio position, when combined with the evolution of returns that are driven by relative wealth dynamics, allows for gains from intertemporal trade in the economy with derivatives, even though there are no gains when only real bonds can be traded. Take the position $\theta = 0.5$, where the two countries have exactly equal net wealth, and given the symmetry in the model, the current account of each country is zero. Say that there is a rise in W_f, driven for instance by a positive technology shock in the foreign country. This will raise θ. If there were trade only in a real risk-free bond, this would simply permanently increase the foreign country's expected consumption, and have no impact at all on the home country. But with trade in derivative assets, the rise in θ leads to a fall in R_h and rise in R_f (in the case $\lambda > 0$). This reduces the effective cost of borrowing for the home country, leading it to a higher net foreign debt, higher investment in the domestic technology, and a higher level of wealth and consumption. In this manner, the original positive technology shock in the foreign economy is shared by the home economy. Moreover, we see that there is an essential interrelationship between *net capital flows* and *gross portfolio holdings*. As the home country receives capital inflows when $\theta > 0.5$, it simultaneously increases its borrowing in derivative h, lending in derivative f and investing more in domestic equity. This levered portfolio ensures that its overall cost of borrowing is lowered, facilitating net capital inflows.

2.3.3. Complementarity between derivative assets and real bonds

These results also reveal an interesting feature of the coexistence of risk-free real bonds and derivative bonds. Without derivative bonds, there are no gains from trade in risk-free bonds. But in equilibrium with trade in derivatives, a country experiencing net capital inflows will take a positive position in risk-free bonds. Why is it that risk-free bonds are traded *simultaneously* with derivatives? The key explanation for this is that, with incomplete markets, derivative assets are imperfect vehicles for facilitating capital flows among countries. In effect, derivatives are playing two roles - first by allowing for portfolio diversification, which is important even in the symmetric case where $\theta = 0.5$ and each country's NFA position is zero. But when $\theta > 0.5$, ($\theta < 0.5$), countries have non-zero NFA positions, and current accounts are imbalanced. In the absence of real risk-free bonds, this would be facilitated by imbalanced movements (in opposing directions) of derivative assets. When the foreign country grows larger ($\theta > 0.5$), the home country must go into a negative NFA position in order to invest in its own technology. But its position in the real risk-free bond may be positive or negative. For $\lambda > 0$, the above results show that it takes a net negative position in derivative assets, which involves a larger negative position in derivative h than its positive position in derivative f. In this case, it may be a debtor or creditor in real risk-free assets, depending on the sign of $\Delta - \lambda\sigma$. With low Δ, the intrinsic risk of derivative assets is low, and the home country takes larger gross positions in the derivatives, and its gross position in derivative h exceeds that of derivative f. In this case, due to the fact that derivative h covaries positively with its own real technology, the portfolio diversification motive more than suffices to facilitate capital flows to invest in its domestic technology. The home country therefore balances this by taking a positive position in real risk-free bonds. On the other hand, when Δ is large, so that $\Delta - \lambda\sigma > 0$, the intrinsic risk of derivative assets reduces their use in capital flows, and the country will optimally choose to issue (so a negative position) risk-free

[9] This logic is similar to the effect of country size on asset returns explored by Yu (2015), but Yu (2015) focuses on country size and financial terms of trade by using a local approximation.

bonds when $\theta > 0.5$. Finally, when $\lambda < 0$, Proposition 3 indicates that the home country will always take a positive net position in derivative assets when $\theta > 0.5$ (with a gross positive in derivative h and negative in derivative f), and balance this by a negative position in real risk-free bonds.

2.4. Conditions for stationarity of θ

So far, we have described θ as a shift variable. But the evolution of θ is determined by endogenous movements in relative wealth levels, driven by productivity and derivative shocks in each country. A fundamental question is whether the wealth distribution is stationary. Thus, while a shock which generates a rise in θ will lead the foreign country to accumulate net claims on the home country, will the rise in θ be self-correcting? For this to be the case, it must be that home wealth grows faster than foreign wealth, when $\theta > 0.5$.

Applying Ito's lemma to Eq. (5) and the equivalent for the foreign country, we may write the diffusion process governing θ as:

$$d\theta = \theta(1-\theta)F(\theta)dt + \theta(1-\theta)G(\theta)dB, \tag{28}$$

where the functional forms of $F(\theta)$, $G(\theta)$, and dB are described in Appendix A. The asymptotic distribution of θ must satisfy either; (a) $\theta \to 1$, (b) $\theta \to 0$, or (c) θ follows a stable distribution in $(0,1)$. Given the form of Eq. (28), clearly $\theta = 1$ and $\theta = 0$ are absorbing states. But the following proposition establishes the conditions under which (c) will apply.

Proposition 2. For $\lambda \neq 0$, and $|\lambda| < 1$, θ has a symmetric ergodic distribution in $(0,1)$ centered at $\theta = \frac{1}{2}$.

Proof. See Appendix B.

The content of this proposition is illustrated through the effect of θ on risk-adjusted growth rates of wealth. The risk-adjusted growth rate for country i as:

$$g_i(\theta) = \lim_{\Delta t \to 0} E_t\left[\frac{\Delta \ln W_i(t+\Delta t)}{\Delta t}\right] = \lim_{\Delta t \to 0} \frac{E_t\left(\frac{\Delta W_i(t+\Delta t)}{W_i(t)}\right) - \frac{1}{2} Var_t\left(\frac{\Delta W_i(t+\Delta t)}{W_i(t)}\right)}{\Delta t}.$$

Then, θ has an ergodic distribution if it cannot access the boundaries 0 or 1. Defining the difference between the foreign and home risk-adjusted growth rate as $\delta(\theta) = g_f(\theta) - g_h(\theta)$, this property holds if the probability of reaching either is zero. For the lower bound, this is the case if $\delta(0) > 0$. Likewise, the probability of reaching the upper bound is zero if $\delta(1) < 0$. This just says that as the home country gets arbitrarily wealthy, relative to the foreign country, the foreign country's risk-adjusted growth rate exceeds that of the home country. Likewise, if the foreign country's wealth increases arbitrarily relative to that of the home country, then the home risk-adjusted growth rate will exceed that of the foreign country.

We may show this directly by computing $\delta(\theta)$. Appendix B shows that $\delta(\theta)$ can be written as:

$$\delta(\theta) = -\frac{(2\theta-1)\sigma^2\lambda^2\left(1-\lambda^2\right)\left(2-\lambda^2\right)}{2\Gamma(\theta)^2} \tag{29}$$

Under the assumptions of the proposition, the denominator of Eq. (29), is always positive, and the numerator is positive (negative) for $\theta < 0.5$ (>0.5). Moreover, this satisfies the conditions

$$\delta(0) = \frac{\sigma^2\lambda^2\left(2-\lambda^2\right)}{2\left(1-\lambda^2\right)} > 0 \tag{30}$$

$\delta(1) = -\delta(0) < 0$, and $\delta(0.5) = 0$. Hence, for $\theta > 0.5$, when the foreign country is relatively wealthy, the home risk-adjusted growth rate exceeds that of the foreign country, and θ falls. The same dynamics occur in reverse when $\theta < 0.5$. The expressions also make clear that the distribution of θ is symmetric. Thus, θ converges towards 0.5 from either direction.

Let us further explore the source of stationarity. Take the case $\lambda > 0$. By Eq. (17), the home (foreign) country issues derivative h (f) to the foreign (home) country. Then by part f) of Proposition 1 the excess return on derivative h relative to derivative f, rises as $\theta > 0.5$. Take the case where a series of positive shocks to the foreign country's technology lead it to increase its share of world wealth, so that θ rises above 0.5. With a higher share of world wealth, the foreign country will demand more derivative h, while issuing more derivative f, as it rebalances its portfolio. Then as described in the previous section, the expected return on derivative h relative to derivative f will be pushed downwards, reducing the cost of borrowing for the home country. Since the expected return on the domestic technology exceeds that on its nominal asset portfolio, this increases the risk-adjusted expected growth rate for the home country, relative to the foreign country. As a result, θ is driven back towards 0.5 again. In effect, it is the levered portfolio composition and its implication for the net borrowing costs for the debtor country as the wealth distribution evolves that ensures the stability of the wealth distribution itself.

While this interpretation is based on a positive value of λ, this is not necessary for the stability result. If $\lambda < 0$, then the equivalent stabilizing force takes place, but now with the home country holding positive positions in derivative h and going short in derivative f. Stability is ensured because it is always the case that countries hold a gross portfolio such that their cost of borrowing falls as the rest of the world gets wealthier.

The stationarity of the wealth distribution here is related to other results in the literature. Many papers have constructed models of bond trading within portfolio choice frameworks - see for instance Engel and Matsumoto (2009), Heathcote and Perri (2013), Coeurdacier et al. (2010) and Coeurdacier and Gourinchas (2016) among others discussed in the introduction. These papers for the most part rely on a two good setup in which endogenous movements in the terms of trade (or real exchange rate) allow for a risk-sharing channel supported by bonds denominated in different goods (or currencies). In this model the real exchange rate is fixed, and diversification is attained by differential covariation of the two derivatives with domestic and foreign technologies. In principle we would expect similar stationarity results to those identified above to hold in a two good setting where in addition of movements in rates of return we would also have variation in the terms of trade. Such an extension however would not admit a simple analytical characterization in the manner presented above.[10]

3. Derivatives interpreted as nominal bonds

As a special case of the model of Section 2 and in order to take the model to the data, we introduce two nominal bonds instead of two derivatives, thereby allowing us to effectively capture the risk-sharing possibilities of bonds denominated in different currencies. We show that the two derivatives described above may be interpreted as nominal bonds issued in home and foreign currencies, and whose returns are subject to inflation risk which covaries differently with home and foreign technologies. Let inflation in country i be represented as,[11]

$$\frac{dP_i}{P_i} = \Pi_i dt + v_i dM_i. \tag{31}$$

Thus, inflation has mean Π_i and standard deviation v_i, which are exogenously given, $i = h$ and f. dM_i represents the increment to a standard Wiener process. The monetary policy followed by country i is represented by the parameters Π_i and v_i, and the covariance of dM_i with dB_i. We let

$$\lim_{\Delta t \to 0} \frac{Cov(\Delta M_i(t+\Delta t), \Delta B_i(t+\Delta t))}{\Delta t} = -\lambda_i, \tag{32}$$

and

$$\lim_{\Delta t \to 0} \frac{Cov\left(\Delta M_i(t+\Delta t), \Delta M_j(t+\Delta t)\right)}{\Delta t} = 0. \tag{33}$$

for $i \neq j$. Eq. (33) here says that inflation shocks are independent across countries. Again, as in the general derivatives case, this is not critical, but simplifies the algebra. The general solution for nominal bonds trade, incorporating real exchange rate shocks (see below) is presented in Appendix D.

Let the nominal value of currency i bonds and the instantaneous return on them be N_i and R_i; that is, $\frac{dN_i}{N_i} = R_i dt$. Suppose that the purchasing power parity still holds as in Section 2. For the home country, the real return on currency i bonds is[12,13]

$$\frac{dN_i}{N_i} - \frac{dP_i}{P_i} = (R_i - \Pi_i)dt - v_i dM_i. \tag{34}$$

[10] A second related literature is that initiated by Schmitt-Grohe and Uribe (2003). They present a quantitative comparison of alternative approaches for generating stationarity in (linearly approximated) small open economy models. Their model with a debt-elastic interest-rate premium is most related to our results. In their small open economy model, the debt distribution will be stationary when the cost of borrowing increases with the size of the net external debt. Intuitively, the incentive to borrow is dampened as the debt level increases. Our model also presents an interesting contrast to Schmitt-Grohe and Uribe (2003). In our two-country general equilibrium setting, the return on gross country liabilities relative to gross assets *falls* as the country becomes more indebted. The key driver of stationarity lies in the excess return to domestic investment relative to external net debt for indebted country. This ensures that indebted countries tend to grow faster and in the long run ensures a stationary world wealth distribution.

[11] We do not explicitly model a source of demand for money. As in Woodford (2003), we can think of the model as representing a 'cashless economy'. What matters is that there is an asset whose payoff depends on the price level, and monetary policy can generate a particular distribution for the price level. An alternative interpretation of this setup is that domestic nominal assets are canceled out exactly by nominal domestic liabilities such as government bonds and central bank notes in each country.

[12] Since we have the single-good world and PPP holds, then the rate of the change in the exchange rate $S\ (= \frac{P_f}{P_h})$ is just determined residually by:

$$\frac{dS}{S} = \left(\Pi_f - \Pi_h + \frac{1}{2}\right)dt + v_f dM_f - v_h dM_h.$$

[13] In this nominal bond equilibrium, long-term bonds are redundant assets and are derivatives of instantaneous nominal bonds. Therefore, given the equilibrium path of instantaneous nominal interest rates, longer-term nominal interest rates are derived completely by arbitrage pricing.

The analogous budget constraint for the home country in the example with nominal bonds may then be written as:

$$\begin{aligned} dW_h &= W_h\left[\omega_T^h(\alpha_h - r) + \omega_h^h(R_h - \Pi_h - r) + \omega_f^h\left(R_f - \Pi_f - r\right) + r\right]dt \\ &\quad - C_h dt + W_h\left(\omega_T^h \sigma_h dB_h - \omega_h^h v_h dM_h - \omega_f^h v_f dM_f\right), \end{aligned} \tag{35}$$

where ω_T^h, ω_h^h, and ω_f^h are the portfolio shares, respectively, of the domestic technology, home currency nominal bonds, and foreign currency nominal bonds. Hence, $1-\omega_T^h-\omega_h^h-\omega_f^h$ represents the share of the real risk-free bond.

It is apparent that the model with nominal bonds is observationally equivalent to the general model with derivative trade, as set out in Sections 2. Hence, propositions 1 and 2 apply exactly as before. Endogenous movements in nominal bond returns on home and foreign currency bonds ensure a stationary world wealth distribution. When $\lambda > 0$, the return on home currency bonds covaries positively with the home technology, and an optimally diversified home portfolio involves the home country issuing home currency bonds and purchasing foreign currency bonds. As the foreign country increases its share of world wealth, the returns on home currency bonds falls relative to that of foreign currency bonds. The parameter v (assuming $v = v_h = v_f$) replaces Δ from the general case. This determines the intrinsic volatility of nominal bonds - higher v will reduce the size of gross bond holdings. As before, agents will also wish to hold real risk-free bonds when $\theta \neq 0.5$.

3.1. Introducing real exchange rate shocks

One drawback of the model under the nominal bond interpretation is that it imposes absolute PPP, so there are no real exchange rate movements. To explore the empirical relevance of the model, we extend the example of the previous section to allow for exogenous real exchange rate shocks, which can be interpreted as being driven by UIP shocks in the spirit of recent papers by Devereux and Engel (2002), Kollmann (2005), Farhi and Werning (2012), Gabaix and Maggiori (2015), Itskhoki and Mukhin (2017) and others. We define $\varepsilon(t)$ as the additional UIP shock. In Appendix C we show that this is equivalent to a temporary departure from PPP for the real exchange rate, and which affects asset returns differentially for home and foreign consumers. Henceforth we refer to it as a 'real exchange rate shock'.

The percent change in real exchange rates ($\frac{d\varepsilon}{\varepsilon}$) deviates temporarily from relative purchasing power parity (PPP) as follows.

$$\frac{d\varepsilon_i}{\varepsilon} = \xi\left(dE_h - dE_f\right), \tag{36}$$

where dE_i represents the increment to a standard Wiener process.[14] $\hat{\xi}^2 = \lim_{\Delta t\to 0}\frac{Var\left(\frac{\Delta\varepsilon_i(t+\Delta t)}{\varepsilon_i(t)}\right)}{\Delta t}$, and $\xi = \frac{\hat{\xi}}{\sqrt{2}}$. Thus, the real exchange rate shock ($\frac{d\varepsilon_i}{\varepsilon_i}$) is zero mean and variance $2\xi^2$ $(= \hat{\xi}^2)$. We let

$$\lim_{\Delta t\to 0}\frac{Cov(\Delta E_i(t+\Delta t), \Delta B_i(t+\Delta t))}{\Delta t} = -\phi_i, \tag{37}$$

and $\lim_{\Delta t\to 0}\frac{Cov\left(\Delta E_i(t+\Delta t), \Delta E_j(t+\Delta t)\right)}{\Delta t} = 0$. for $i \neq j$.

The correlation terms λ_i in Eq. (32) and ϕ_i in Eq. (37) describe the cyclical characteristics of the inflation rate shocks and the real exchange rate shocks, and hence of the real return on nominal bonds. By construction, $0 < |\lambda_i| < 1$, and $0 < |\phi_i| < 1$. In addition, in order to ensure stationarity in θ, we impose the restriction

$$\Phi_i = \left(1-\lambda_i^2\right)\left(v_i^2 + \xi^2\right) + \left(1-\lambda_i^2-\phi_i^2\right)\xi^2 > 0 \tag{38}$$

For the home country, the real return on home currency bonds is the same as Eq. (34) above. But since the real exchange rate deviates temporarily from relative PPP, the real return on foreign currency bonds is determined by

$$\frac{dN_f}{N_f} - \left(\frac{dP_f}{P_f} - \frac{dP_h}{P_h}\right) - \frac{d\epsilon}{\epsilon} - \frac{dP_h}{P_h} = \left(R_f - \Pi_f\right)dt - v_f dM_f - \xi\left(dE_h - dE_f\right).$$

[14] Meese and Rogoff (1983) first documented that nominal exchange rate follows a random-walk-like process, and is not robustly correlated with macroeconomic fundamentals (see Engel and West, 2005). Rogoff (1996) summarized that real exchange rates comove closely with nominal exchange rate at most frequencies, and can be hardly explained by macroeconomic factors (see Chari et al., 2002).

For the foreign country, the situation is the opposite. For foreign consumers, the real return on foreign currency bonds is analogous to (34). But the return on home currency bonds for the foreign consumer is

$$\frac{dN_h}{N_h} - \left(\frac{dP_h}{P_h} - \frac{dP_f}{P_f}\right) + \frac{d\epsilon}{\epsilon} - \frac{dP_f}{P_f} = (R_h - \Pi_h)dt - v_h dM_h - \xi\left(dE_f - dE_h\right).$$

Again, to highlight the role of nominal bonds in fostering intertemporal trade, we will focus on the case where countries have identical drift and diffusion parameters, so that, $\alpha_h = \alpha_f = \alpha$, $\sigma_h = \sigma_f = \sigma$, $\Pi_h = \Pi_f = \Pi$, $v_h = v_f = v$, $\lambda_h = \lambda_f = \lambda$, $\phi_h = \phi_f = \phi$, and therefore $\Phi_h = \Phi_f = \Phi$.[15]

We can also solve for equilibrium returns following the same steps as in Section 2. Appendix D derives the solutions for equilibrium returns and portfolio holdings in this case with nominal bonds and real exchange rate shocks. For the equilibrium returns at the point of equal national wealth levels ($\theta = 0.5$),

$$\bar{r} = \alpha - \left[1 - \frac{(\lambda v - \phi\xi)^2}{2\left(v^2 + \xi^2\right)}\right]\sigma^2, \tag{39}$$

$$\overline{R}_h = \overline{R}_f = \overline{R} = \bar{r} + \Pi + \frac{\lambda\sigma v\left(v^2 + 2\xi^2\right) + \phi\sigma v^2\xi}{2\left(v^2 + \xi^2\right)}. \tag{40}$$

Note that $\bar{r} < \alpha$, so the maximized risk-free rate is less than α, and again, neither country takes short positions in its domestic technology. In addition, we see that real exchange rate risk affects equilibrium returns due to the fact that the real exchange rate shock covaries with the domestic and foreign technology.

The equilibrium portfolio holdings in the case of $\theta = 0.5$ are now written as

$$\overline{\omega}_T^h = 1, \tag{41}$$

$$\overline{\omega}_h^h = -\overline{\omega}_f^h = -\frac{\sigma(\lambda v - \phi\xi)}{2\left(v^2 + \xi^2\right)}. \tag{42}$$

Thus, as before, the net positions in both nominal bonds ($\overline{\omega}_h^h + \overline{\omega}_f^h$) and real bonds ($1 - \overline{\omega}_T^h - \overline{\omega}_h^h - \overline{\omega}_f^h$) are zero at $\theta = 0.5$. But the optimal portfolio diversification now depends not just on inflation risk, but also on real exchange rate risk. For $\lambda v - \phi\xi > 0$, the home country will held a short position in home currency bonds, balanced by an equal long position in foreign currency bonds. Intuitively, even if $\lambda < 0$, so that home bonds would, ceteris paribus, represent a good hedge against domestic technology shocks, it may still be optimal to diversify away from home bonds towards foreign bonds, since the movement of the real exchange rate, which affects the return on foreign bonds, offers a better hedge.

The solutions for returns and portfolio holdings in the general case where $\theta \neq 0.5$ are presented in Appendix D. There it is shown that the analogue of Proposition 1 holds under the conditions set out above and some additional weak assumptions on parameters. The following propositions summarize the results, whose proofs are delegated to Appendix D.

Proposition 3. *In the equilibrium with trade in nominal bonds,*

a) The real risk-free interest rate is always above the autarky level ($r^A = \alpha - \sigma^2$), and below α. But, it may exceed even the complete markets level ($r^C = \alpha - \frac{1}{2}\sigma^2$) in the neiborhood of $\theta = 0.5$.

b) For $\lambda v - \phi\xi > 0$ and $(1 - \lambda^2)v^2 - \lambda\phi v\xi > 0$, each country holds a short position in its own-currency nominal bonds, and a long position in the other currency nominal bonds.

c) For $\lambda v - \phi\xi > 0$, the home country holds a positive (negative) net position in nominal bonds for $\theta < 0.5$ ($\theta > 0.5$).

d) The home country holds a positive or negative share in real risk-free bonds given θ.

e) Let $\rho(\theta) = R_h(\theta) - R_f(\theta)$ be defined as the risk-premium on home-currency relative to foreign-currency nominal bonds. Then when $\lambda v - \phi\xi > 0$, $\rho(\theta)$ is positive (negative) for $\theta < 0.5$, ($\theta > 0.5$).

[15] None of the results are qualitatively different when drift and diffusion parameters differ across countries.

f) For $\lambda\nu - \phi\xi > 0$ and $\lambda(\nu^2 + 2\xi^2) - \nu\phi\xi > 0$, the home country has a positive (negative) net foreign asset (NFA) position $(1 - \omega_T^h)$ as $\theta < 0.5$, $(\theta > 0.5)$.

Proposition 4. For $\lambda\nu \neq \phi\xi$, and $\Phi = (1 - \lambda^2)(\nu^2 + \xi^2) + (1 - \lambda^2 - \phi^2)\xi^2 > 0$, θ has a symmetric ergodic distribution in $(0,1)$ centered at $\theta = 0.5$.

Let us assume $\lambda\nu - \phi\xi > 0$. In the analogous case to Section 2 with $\lambda > 0$, each country holds a short position in its own-currency nominal bonds, and a long position in the other currency nominal bonds. In addition, the home country holds a positive (negative) net position in both nominal bonds $(\omega_h^h + \omega_f^h)$ and net foreign assets $(1 - \omega_T^h)$ for $\theta < 0.5$ $(\theta > 0.5)$. Finally, defining as before $\rho(\theta) = R_h(\theta) - R_f(\theta)$ as the risk-premium on home-currency relative to foreign-currency nominal bonds, we show that $\rho(\theta)$ is positive (negative) for a positive eternal position $(\theta < 0.5)$, (a negative eternal position, $\theta > 0.5$). That is, a debtor country enjoys lower borrowing rates and higher lending rates. Then, as before, if $\lambda\nu - \phi\xi < 0$ (along with the other conditions above), the portfolio position of the home country will be reversed, but the stationarity of the wealth distribution will still obtain, through the endogenous movement in asset returns as a function of θ. Thus, the essential channel of the previous section, ensuring that the net cost of borrowing is lower for debtor countries and higher for creditor countries, holds in this extended model with real exchange rate risk.

3.2. Empirical evidence

Here we explore the extent to which there is empirical support for the model, and in particular for the version of the model with nominal bond trade and UIP or real exchange rate shocks. First, we acknowledge that with log preferences and conventional productivity shocks, there is little possibility of generating substantial risk-premia in this model. But the model also brings predications about the configuration of external portfolios. We now provide some evidence along this dimension.

3.2.1. The sign of $\lambda\nu - \phi\xi$

The results described in the previous section imply that as long as $\lambda\nu - \phi\xi \neq 0$ and $\Phi > 0$, the self-correcting mechanism of external imbalances holds, but the sign of $\lambda\nu - \phi\xi$ will determine the currency positions of external assets and liabilities, and also the excess returns on external assets and liabilities. This section provides illustrative estimates of these parameters. The model is mapped to the data as follows. Q_i is interpreted as real stock market price index in a country, P_i as consumer price index, and ε as effective real exchange rate. Given the data availability, we collect quarterly data for G-7 countries and seven other major economies including Netherlands, Australia, Mexico, Brazil, China, India and Russia over the period 1999:1–2017:4. We treat each country in the data sample as the home country and the trade-weighted aggregate of all the other countries as the foreign country. The data sources and constructions are described in the online Data Appendix. Table 1 presents the estimated value of $\lambda\nu - \phi\xi$. The results show that $\lambda\nu - \phi\xi > 0$ for most of country-pairs in the data sample. This represents the main necessary condition required for each country to issue its net external liabilities in its own currency.[16]

One caveat is that while the model implies that the sign of $\lambda\nu - \phi\xi$ will affect the currency composition of external assets and liabilities, it does not capture the different risk categories of assets and liabilities in the national balance sheet, which may be important, in reality, for assessing the risk sharing capacities of the external asset and liabilities position. In the following empirical analysis, we aggregate all kinds of bond assets or liabilities on a country's balance sheet as the counterpart of the model.

3.2.2. The empirical relationship between excess returns and net foreign asset positions

A key prediction of the model is that debtor countries will face lower real returns on their nominal liabilities than they receive on their nominal assets. In our model, the excess return on home currency bonds is given by $\rho(\theta) = R_h(\theta) - R_f(\theta) = -\frac{\Phi}{\Psi(\theta)}\sigma\nu^2(2\theta-1)(\lambda\nu-\phi\xi)$. The net foreign assets of the home country relative to epected GDP are $\frac{1-\omega_T^h}{\alpha\omega_T^h}$.[17] Since the stationary distribution of wealth is symmetric and centered at $\theta = 0.5$, then foreign assets of the home country relative to the *average* GDP equals $\frac{1-\omega_T^h}{\alpha} = -\frac{1}{\alpha\Psi(\theta)}\theta(2\theta-1)\nu(\lambda\nu-\phi\xi)\left[\lambda\left(\nu^2+2\xi^2\right)-\phi\xi\right]$. When the home country is a net foreign debtor $(\theta > 0.5)$, $\rho(\theta) < 0$ and home is short in home currency denominated debt; when the home country is a net foreign creditor, the opposite relationship holds.

A large literature has estimated the return differentials between external assets and liabilities, and in particular many papers show that the United States enjoys positive excess returns of its external assets over liabilities, even though it has a large negative net foreign asset position (see for instance, Gourinchas and Rey, 2007b; Lane and Milesi-Ferretti, 2007a; Curcuru et al., 2008; Forbes, 2010).[18] The ratio of US net foreign assets to GDP has a significantly negative effect on the excess return received by the rest of the world on US investments.

[16] As shown in the online Data Appendix, $(1-\lambda^2)\nu^2 - \lambda\phi\nu\xi > 0$ and $\lambda(\nu^2 + 2\xi^2) - \nu\phi\xi > 0$, which are required by Proposition 3 (b) and (f) are satisfied in most cases. $\Phi > 0$ is satisfied in all cases.

[17] Here, domestic production is assumed to be determined according to *Ak* type technology.

[18] The return differentials can be further decomposed into the composition effect (Gourinchas and Rey (2007a)), return effect (Gourinchas and Rey (2007a); Curcuru et al., 2013), and timing effect (Curcuru et al., 2010). More recent reviews can be found in Curcuru et al. (2013) and Gourinchas and Rey (2014).

Table 1
The value of $\lambda v - \phi\xi$ in the data from 1990:1–2017:4.

	AUS	BRA	CAN	CHN	DEU	FRA	GBR
Whole sample	0.029	0.090	0.013	0.002	0.006	−0.005	0.012
Before 2007	0.024	0.115	0.011	0.003	−0.000	−0.011	0.006
After 2007	0.033	0.048	0.014	0.001	0.015	0.005	0.017
	IND	ITA	JPN	MEX	NLD	RUS	USA
Whole sample	0.009	−0.001	0.010	−0.009	−0.005	0.053	−0.002
Before 2007	0.005	−0.006	0.015	−0.014	−0.009	0.065	0.001
After 2007	0.016	0.006	0.001	−0.001	0.004	0.038	−0.010

Does the same phenomenon appear in observations for other countries? Based on the Balance of Payments and International Investment Positions for a group of 19 advanced countries (excluding major financial centers) and 32 emerging and developing countries during 1980 − 2016, we construct the gross returns on external assets and liabilities as

$$r_t^R = \frac{A_t^R - A_{t-1}^R - FLOW_t^R}{A_{t-1}^R} + \frac{INC_t^R}{A_{t-1}^R} \tag{43}$$

where A_t^R is the position (assets or liabilities) at the end of period t, $FLOW_t^R$ represents financial flows during period t and INC_t^R stands for investment income.[19] Fig. 1 shows that the excess returns of external assets over liabilities are negatively associated with net foreign debt asset positions. The return differentials faced by emerging and developing economies seem to respond more aggressively to their net foreign debt asset positions than those of the advanced economies. Fig. 2 shows that when we include all categories of assets and liabilities, there is still a negative association between excess returns and net foreign asset positions.[20] This is consistent with our model, which states that countries with higher net foreign asset positions tend to have lower excess returns of their external assets over liabilities.

4. Extensions of the basic model

4.1. No trade in real bonds

In reality, almost all international bond trade is carried out with nominal bonds. If we restrict the model so that only nominal bonds are traded, the essential results are unchanged. To solve the model in this case, we impose Eqs. (7) and (8) in combination with the restrictions of zero supply of real bonds within each country; so that $\omega_T^h + \omega_h^h + \omega_f^h = 1$, and $\omega_T^f + \omega_h^f + \omega_f^f = 1$. To simplify the exposition in this case, we return to the simplified model without real exchange rate shocks ($\xi = 0$).

In the case where there is no trade in real risk-free bonds, the solution for portfolio holdings is then

$$\omega_h^h = -\frac{\theta\lambda\sigma\left(v^2 + 2\sigma^2 - 2\lambda\sigma v - \lambda\sigma v(1-2\theta)\right)}{v\left(v^2 + 2\sigma^2 - 2\lambda\sigma v - \sigma^2\lambda^2(1-4\theta(1-\theta))\right)}, \tag{44}$$

and

$$\omega_f^h = \frac{\theta\lambda\sigma\left(v^2 + 2\sigma^2 - 2\lambda\sigma v + \lambda\sigma v(1-2\theta)\right)}{v\left(v^2 + 2\sigma^2 - 2\lambda\sigma v - \sigma^2\lambda^2(1-4\theta(1-\theta))\right)}. \tag{45}$$

From the above equations, $\omega_h^h < 0$ (>0), as $\lambda > 0$, (< 0), and $\omega_f^h > 0$ (<0), as $\lambda > 0$, (<0) in the neighborhood of $\theta = 0.5$.[21] Thus, part b) of Proposition 3 applies as before. Then adding Eqs. (44) and (45) together we get

$$\omega_h^h + \omega_f^h = -\frac{2\theta(2\theta-1)\lambda^2\sigma^2}{v^2 + 2\sigma^2 - 2\lambda\sigma v - \sigma^2\lambda^2(1-4\theta(1-\theta))}, \tag{46}$$

[19] BoP and IIP data are imbalanced during 1980 − 2016. Note that this measure can be criticized along a number of dimensions. The main criticism is that the procedures for constructing international investment positions generally followed by national statistical agencies do not always enforce consistency between the net IIP entries and the current account entries. Another issue concerns the calculation of the valuation effects. The literature makes a number of different assumptions to deal with data consistency and errors and omissions in the BoP and IIP tables. In the baseline case, we use the raw data, since we focus on the association of net foreign asset positions and excess returns of external assets over liabilities. As a robustness check, we also used the data from Lane and Milesi-Ferretti (2007b) and Habib (2010). The results were very similar to those shown here, see also the online Data Appendix.

[20] We also explored the associations for periods before and after the Global Financial Crisis. The patterns were very similar (see the online Data Appendix.)

[21] Note that $v^2 - 2\lambda\sigma v + 2\sigma^2 - \sigma^2\lambda^2 > (v-\sigma)^2 + (1-\lambda^2)\sigma^2 > 0$ as long as $|\lambda| < 1$.

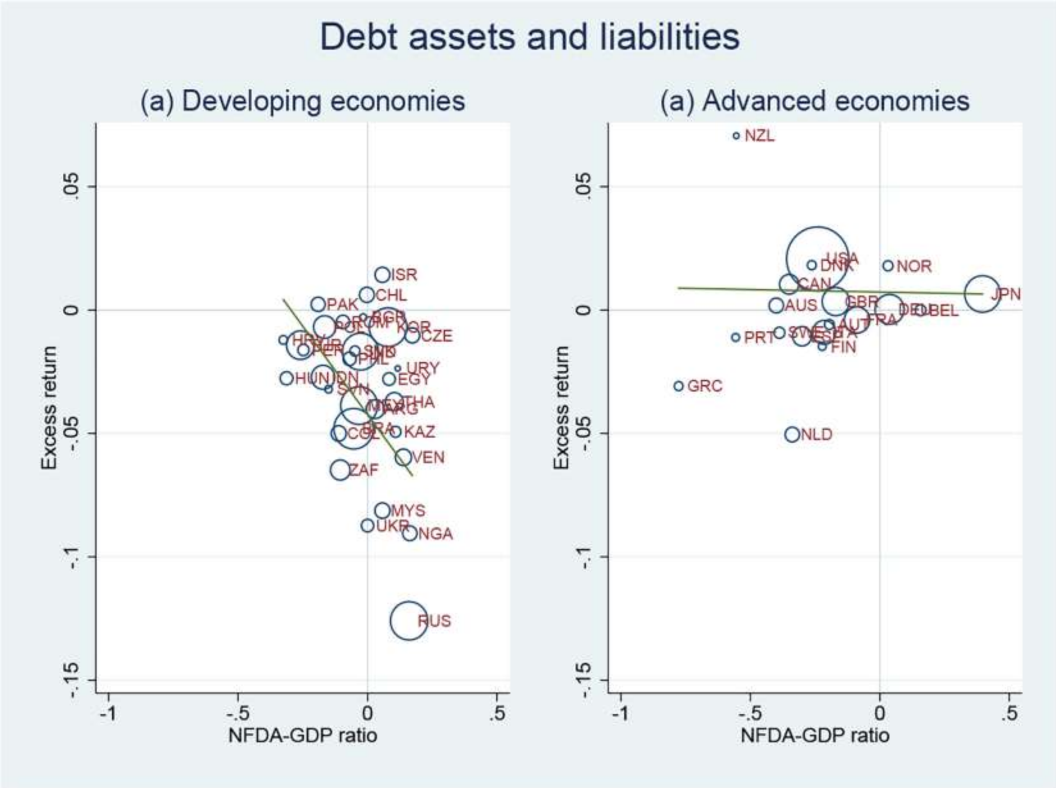

Fig. 1. Excess returns of external debt assets over external debt liabilities and net external debt asset positions over GDP (NFDA-GDP ratio) for a group of advanced economies and developing economies over 1980 − 2016. Note: Both variables are in level. The return on external assets (liabilities) is defined as the ratio of investment income and capital gains to the corresponding external asset (liability) positions. Investment income are taken from the Balance od Payments and capital gains and positions are from the International Investment Positions of IMF database. "Debt" assets (liabilities) are defined as portfolio debt investment, other investment and reserves net of gold assets (liabilities) in BoP and IIP tables. Bubbles are proportional to real GDP in 2006. The dark green solid line in the figure is the linear fitted line weighted by real GDP.

which establishes the equivalent of part e) of proposition 3.[22]

The risk premium on home currency bonds may now be written as:

$$\rho(\theta) = -\frac{\left(v^2 + 2\sigma^2 - 2\lambda\sigma v - \sigma^2\lambda^2\right)(2\theta - 1)\lambda\sigma v}{v^2 + 2\sigma^2 - 2\lambda\sigma v - \sigma^2\lambda^2(1 - 4\theta(1-\theta))}. \tag{47}$$

Again, this is negative (positive) as $\lambda > 0$, ($\lambda < 0$), for $\theta > 0.5$, and vice versa. Hence, part f) of Proposition 3 holds as before.

The only difference between this case and the benchmark model above is that all trade must be intermediated by nominal bonds. As a country experiences capital inflows, these must be all financed by issuing domestic currency bonds (for $\lambda > 0$), but hedged by also purchasing foreign currency bonds. Again, the risk-premium evolves so that the return on gross liabilities of a debtor country are below the return it receives on its gross assets.

Since the movements in the risk-premium are qualitatively as before, the stationarity result of Proposition 4 holds in the same way as before. Using the same technique in Appendix B, we may write $\delta(\theta)$ as:

$$\delta(\theta) = \frac{\lambda^2\sigma^2(1-2\theta)\left(v^2 - 2\lambda\sigma v + 2\sigma^2 - \sigma^2\lambda^2\right)\left(v^2 - 2\lambda\sigma v + 2\sigma^2\right)}{\left(v^2 + 2\sigma^2 - 2\lambda\sigma v - \sigma^2\lambda^2(1 - 4\theta(1-\theta))\right)^2}.$$

This satisfies $\delta(0.5) = 0$, and $\delta(\theta) > 0$ (< 0) for $\theta < 0.5$ (> 0.5), as long as $\lambda \neq 0$. Thus, as before, the relatively poorer country will be a net foreign debtor, but grows faster than the richer country, ensuring a stable distribution of world wealth.

[22] There is a subtle difference from the model presented in Section 3. As a numerator of the right hand side of Eq. (46) shows, the behavior of net nominal bond positions (equivalent to net foreign asset positions) no longer depends on the sign of λ.

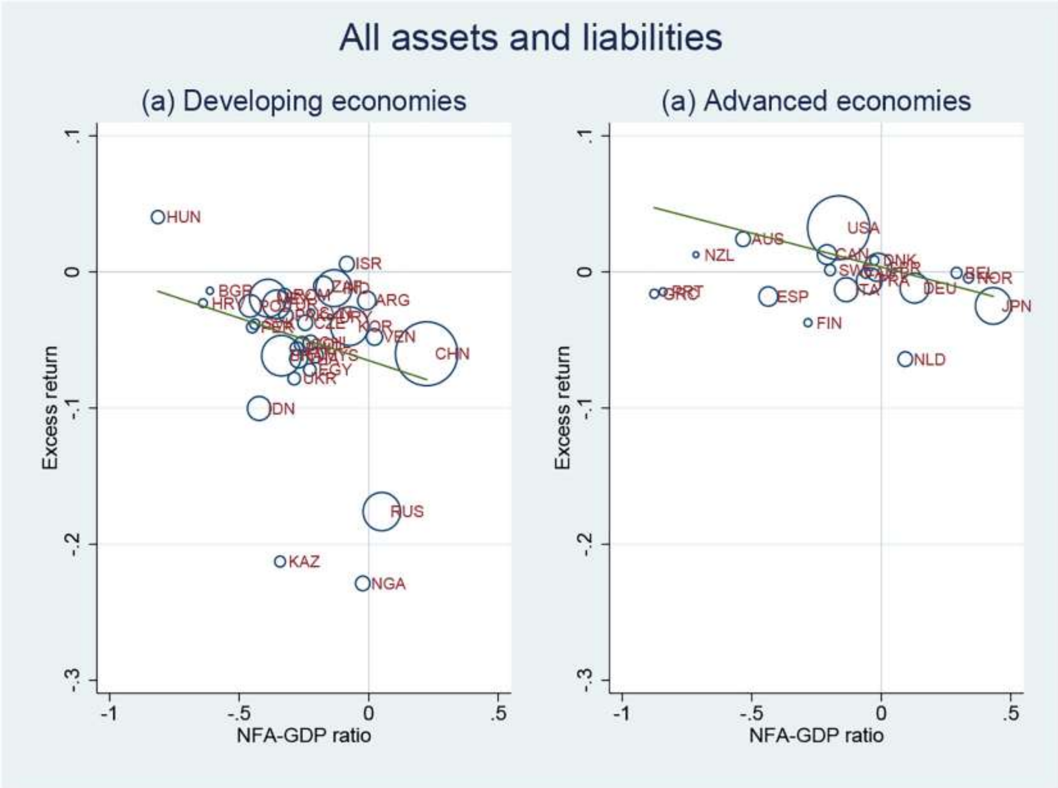

Fig. 2. Excess returns of external assets over external liabilities and net external asset positions over GDP (NFA-GDP ratio) for a group of advanced economies and developing economies over 1980 − 2016. Note: Both variables are in level. The return on external assets (liabilities) is defined as the ratio of investment income and capital gains to the corresponding external asset (liability) positions. Investment income are taken from the Balance od Payments and capital gains and positions are from the International Investment Positions of IMF database. Bubbles are proportional to real GDP in 2006. The dark green solid line in the figure is the linear fitted line weighted by real GDP.

4.2. One-way capital flows

It is widely recognized that many countries can not or do not issue debt denominated in their own currency (e.g. Lane and Shambaugh (2010a, 2010b); Bordo et al. (2010). In fact, much of the nominal debt traded internationally is denominated in US dollars. We now briefly look at another special case of the model which captures this phenomenon. We restrict all trade in nominal bonds to take place in the home currency only. Even if $\lambda > 0$, the foreign country cannot issue its own currency debt.

For simplicity assume that there is no trade in real risk-free bonds. In addition, as in the previous subsection, we abstract from real exchange rate shocks. Optimal portfolio rules (ω_T^h, ω_h^h, ω_T^f, and ω_h^f) are still determined by a version of Eq. (10) with $\lambda_h = \lambda > 0$ for the home country choice and $\lambda_f = 0$ for the foreign country. Then, two portfolio restrictions ($\omega_T^h + \omega_h^h = 1$, $\omega_T^f + \omega_h^f = 1$) and a bond market clearing ($(1-\theta)\omega_h^h + \theta\,\omega_h^f = 0$) may be used to determine the equilibrium nominal interest rates on home currency bonds (R_h), and the real risk-free interest rate on implicit (non-traded) real bonds (r_h and r_f). The home country's holding of home bonds is given by

$$\omega_h^h = \frac{-\lambda\theta v\sigma}{-2\lambda\theta v\sigma + v^2 + \sigma^2}. \tag{48}$$

As before, the home country has a negative position in home currency bonds, when $\lambda > 0$. The difference now however is that Eq. (48) represents both the *gross* and *net* bond position of the home country. When $\lambda > 0$, the home country *always* has a negative net foreign asset position. The international capital market is asymmetric in structure. To hedge against domestic consumption risk, the home country wishes to issue domestic denominated bonds. The foreign country is willing to purchase these bonds because their returns are uncorrelated with foreign technology shocks.

To further gain insight into this example, restrict attention to the case $v = \sigma$. Then, the nominal interest rate on the home currency bond is:

$$R_h = \alpha + \pi - \frac{\sigma^2(1-\lambda)}{1-\lambda\theta}.$$

This is declining in θ, for $\lambda > 0$. Thus, the return on home net foreign liabilities falls as the foreign country gets relatively wealthier. This ensures that the same stationarity condition holds as before. We may calculate the $\delta(\theta) = g_f(\theta) - g_h(\theta)$ function as follows:

$$\delta(\theta) = \frac{\sigma^2\lambda^2(1-\theta(2-\lambda\theta))}{4(1-\lambda\theta)^2}. \tag{49}$$

This satisfies the conditions $\delta(0) > 0$, and $\delta(1) < 0$, for $\lambda \neq 1$. But unlike the symmetric economy, we now have $\delta(0.5) = \frac{1}{4}\frac{\sigma^2\lambda^3}{(2+\lambda)^2}$, which is positive for $\lambda > 0$. Thus, the long run wealth share is not equalized across countries. We may use Eq. (49) to establish that the unconditional mode of θ is $\theta = \frac{-1-\sqrt{1-\lambda}}{\lambda}$, which exceeds 0.5 for $\lambda < 0$. If the home country is a net foreign debtor in its own currency, then the long run distribution of world wealth is skewed in favor of the foreign country. Moreover, the higher in absolute value is λ, the higher the foreign country's long run share of world wealth. Unlike the symmetric world economy where bonds of either currency can be traded internationally, in the case where only a single currency bond is acceptable, the debtor country achieves risk sharing only by accepting a lower and lower share of world wealth.

4.3. Substitutability between nominal bonds and equity trade

This subsection relaxes the assumption that shares in the national production technologies are non-tradable across countries. We extend the model to allow for trade in bonds and partial trade in shares in production technologies (or equities). If equities were freely traded, then financial markets would be complete, and each country would hold a perfectly diversified equity portfolio with half their portfolio in home equity and half in foreign equity. But it is reasonable to assume that there is some part of each country's production technology that is not traded internationally.[23] Assume now that there are two linear production technologies in each country. As before, the technology described by Eq. (1) is not tradable internationally. But there another technology characterized as

$$\frac{dQ_i^E}{Q_i^E} = \beta dt + \epsilon dB_i^E, \tag{50}$$

for $i = h$ or f, where dB_i^E is an increment to the standard Wiener process uncorrelated with dB_i, and correlated with dM_i with the coefficient $-\lambda$.[24]

Assume that shares in the technology described by Eq. (50) are tradable in each country. Now, in addition to investments in its own non-tradable technology ω_T^h (ω_T^f), the home (foreign) country can invest in its own tradable technology with a portfolio weight ω_{Th}^h (ω_{Tf}^f), and the tradable technology of the foreign (home) country with a portfolio weight ω_{Tf}^h (ω_{Th}^f).

Holding a share in technology in this model is equivalent to making a direct investment in the production technology. Thus, we impose a restriction that investors cannot take a short position on the tradable technologies. Then, the following portfolio restrictions must be satisfied for both countries: $\omega_T^h + \omega_{Th}^h + \omega_{Tf}^h + \omega_h^h + \omega_f^h = 1$ with $\omega_T^h \geq 0$, $\omega_{Th}^h \geq 0$, and $\omega_{Tf}^h \geq 0$ for the home country, and $\omega_T^f + \omega_{Tf}^f + \omega_{Th}^f + \omega_f^f + \omega_h^f = 1$ with $\omega_T^f \geq 0$, $\omega_{Tf}^f \geq 0$, and $\omega_{Th}^f \geq 0$ for the foreign country.

To illustrate the properties of this extended model, we make the additional assumptions; $\alpha = \beta$, and $\sigma = \varepsilon = \nu$. These assumptions are not essential, but help to simplify the exposition.

The model can be solved in the same manner as before. The portfolio holdings and returns depend on the state variable θ. The model is still entirely symmetric, so that countries have zero NFA at $\theta = 0.5$. The nominal bond portfolio for the home country at $\theta = 0.5$, is given by $\omega_h^h = -\omega_h^f = \frac{-\lambda}{2(3-4\lambda^2)}$. As in the previous case without direct trade in shares of the production technologies, the home (foreign) country still takes a short position in the home currency bond, and a long position in the foreign currency bond, when λ is positive, and $\lambda^2 \leq \frac{1}{2}$. The condition $\lambda^2 \leq \frac{1}{2}$ now defines the range of λ for which markets are effectively incomplete (see below).

At the point $\theta = 0.5$, shares in the home non-tradable technology, and the home and foreign tradable technologies are given by $\omega_T^h = \frac{1-\lambda^2}{3-4\lambda^2}$, $\omega_{Th}^h = \frac{1-\lambda^2}{3-4\lambda^2}$, and $\omega_{Tf}^h = \frac{1-2\lambda^2}{3-4\lambda^2}$. These are all non-negative under the condition $\lambda^2 \leq \frac{1}{2}$. In addition,we confirm that $\omega_T^h + \omega_{Th}^h + \omega_{Tf}^h = 1$ in this case, so NFA is indeed equal to zero.

Under this parameterization and $\theta = 0.5$, the home and foreign bond market clearing condition determine equilibrium interest rates equal to $R_h = R_f = \alpha + \Pi - \frac{[2-\lambda(3+3\lambda-4\lambda^2)]\sigma^2}{6-8\lambda^2}$, while the corresponding risk-free rates (r_h and r_f) are equal to $\alpha - \frac{(2-3\lambda^2)\sigma^2}{6-8\lambda^2}$. At the lower limit of $\lambda^2 = 0$, nominal bonds play no role in hedging consumption risk, and an equilibrium is characterized by each country dividing its wealth equally over the three technologies (the domestic non-tradable and two tradable technologies). The

[23] For instance, evidence for this is presented in Kho et al. (2009). An alternative would be to allow for trading costs of diversification in the single equity used in most of the paper, but this would lose the analytical tractability of the present model. Here, we do not mean to explain the home equity bias observations but simply to illustrative the effects of loosening of the trading restrictions in the main part of the paper.

[24] An indirect correlation between dB_i^E and dB_i does not show up at the variance-covariance matrix because it converges to zero as $\Delta t \to 0$ by a higher-order effect.

equilibrium risk-free rate is then equal to $\alpha-\frac{1}{3}\sigma^2$. As long as λ is non-zero, nominal bond trading can still play an effective role in sharing country-specific shocks, even when there is tradable equity.[25]

Again, θ has a stationary distribution in $(0,1)$ centered at $\theta=0.5$. As before, define $\delta(\theta)=g_f(\theta)-g_h(\theta)$. It is possible to show that $\delta(0.5)=0$, and

$$\delta(1)=-\delta(0)=-\frac{\left(2-3\lambda^2\right)\sigma^2\lambda^2}{18\left[1-3\lambda^2\left(1-2/3\lambda^2\right)\right]}<0,$$

as long as $0<\lambda^2\le\frac{1}{2}$. Therefore, when a country's share of world wealth falls, its relative growth rate increases, ensuring stationarity of θ. Despite the ability to trade equity, the underlying force behind the stationarity condition is the presence of nominal bond trading, just as in the previous case.

We saw above that trade in nominal bonds was complementary to trade in real risk-free bonds. But nominal bonds may be substitutable for trade in equity. Even in the case where equity is tradable, cross country equity holdings may be small. In particular, as $\lambda^2\rightarrow\frac{1}{2}$, we find that $\omega^h_{Tf}\rightarrow 0$, $\omega^h_T\rightarrow\frac{1}{2}$, and $\omega^h_{Th}\rightarrow\frac{1}{2}$. Thus, as the nominal bond markets become more proficient at risk sharing, the direct holding of foreign equity goes to zero, and home agents hold 100% of the home technologies (both non-tradable and tradable). Thus, although direct trade in equity is possible, the portfolio equilibrium is characterized by complete *home bias* in equity holdings. The intuitive reason for this is that the bond portfolio held by residents of each country represents a perfect claim on the foreign technology in the case when $\lambda^2\rightarrow\frac{1}{2}$. Thus, our initial assumption that there is no trade in equity becomes an *equilibrium outcome*, the better the risk-sharing characteristics of nominal bonds.[26]

5. Conclusion

This paper is mainly concerned with developing a simple framework for understanding the interaction between portfolio dynamics and current account dynamics within an incomplete markets general equilibrium model. The model is attractive in that it allows for a complete analytical characterization of time-varying portfolio shares and returns, as well as an analytical description of the world distribution of wealth. The main message is that external imbalances could be endogenously adjusted through time-varying asset returns. As Section 4 shows, the model can easily be adapted to allow for multiple types of assets and constraints, and endogenous investment and production as in Brunnermeier and Sannikov (2015). The key underlying feature of the model is that a stationary distribution of net foreign assets is guaranteed by time variation in the return on nominal bonds.

In future work, we plan to investigate more fully the empirical implications of the model, as well as extending the model to allow for differences in growth rates and volatility in technologies among countries.

Appendix

Appendix A. Process of Wealth Distribution θ

To obtain the process of wealth distribution θ $\left(=\frac{W_f}{W_h+W_f}\right)$, we define $m_h(\theta)=\lim_{\Delta t\rightarrow 0}\frac{E_t\left[\frac{\Delta W_h(t+\Delta t)}{W_h(t)}\right]}{\Delta t}$, $m_f(\theta)=\lim_{\Delta t\rightarrow 0}\frac{E_t\left[\frac{\Delta W_f(t+\Delta t)}{W_f(t)}\right]}{\Delta t}$, $n_h(\theta)=\lim_{\Delta t\rightarrow 0}\frac{Var_t\left[\frac{\Delta W_h(t+\Delta t)}{W_h(t)}\right]}{\Delta t}$, $n_f(\theta)=\lim_{\Delta t\rightarrow 0}\frac{Var_t\left[\frac{\Delta W_f(t+\Delta t)}{W_f(t)}\right]}{\Delta t}$, and $n_{hf}(\theta)=\lim_{\Delta t\rightarrow 0}\frac{Cov_t\left[\frac{\Delta W_h(t+\Delta t)}{W_h(t)},\frac{\Delta W_f(t+\Delta t)}{W_f(t)}\right]}{\Delta t}$. Then, using Ito's lemma, we can derive the process of wealth distribution θ $\left(=\frac{W_f}{W_h+W_f}\right)$ as.

$$d\theta=\theta(1-\theta)[F(\theta)dt+G(\theta)dB], \tag{A.1}$$

where

$$\begin{aligned}F(\theta)&=m_f(\theta)-m_h(\theta)-\theta n_f(\theta)+(1-\theta)n_h(\theta)+(2\theta-1)n_{hf}(\theta),\\ G(\theta)&=\sqrt{n_h(\theta)+n_f(\theta)-2n_{hf}(\theta)},\end{aligned}$$

[25] At the upper limit of $\lambda^2=\frac{1}{2}$, on the other hand, the equilibrium risk-free rate reaches $\alpha-\frac{1}{4}\sigma^2$, which is equivalent to the risk-free rate in the complete markets case where each country divides its wealth equally over the two domestic and two foreign technologies.

[26] Coeurdacier and Gourinchas (2016) present a model where trade in bonds represents a substitute for equity trade. Their result hinges on endogenous movements in the terms of trade in a multi-good environment, while here, it is time varying differences in the risk-sharing capacity of nominal bonds that is key to the substitutability between bonds and equity trade.

and

$$dB = \frac{1}{G(\theta)}\left[\omega_T^f(\theta)\sigma dB_f - \omega_1^f(\theta)\Delta_a dD_1 - \omega_1^f(\theta)\Delta_b dD_2\right] - \frac{1}{G(\theta)}\left[\omega_T^h(\theta)\sigma dB_h - \omega_1^h(\theta)\Delta_a dD_1 - \omega_2^h(\theta)\Delta_b dD_2\right].$$

$dB(t)$ is newly defined as the increment to a standard Brownian motion. Note here that

$$\lim_{\Delta t\to 0}\frac{E_t[\Delta B(t+\Delta t)]}{\Delta t} = 0, \quad \lim_{\Delta t\to 0}\frac{Var_t[\Delta B(t+\Delta t)]}{\Delta t} = 1.$$

Appendix B. Stationarity of Wealth Distribution θ

To make theorems 16 and 18 of Skorokhod (1989) applicable, we consider the process of κ or $\ln\frac{\theta}{1-\theta}$ $(= \ln\frac{W_f}{W_h})$ instead of θ. The process of κ is derived as.

$$d\kappa = \delta(\theta)dt + G(\theta)dB, \tag{B.1}$$

where $\theta = \frac{\exp(\kappa)}{1+\exp(\kappa)}$, and $\delta(\theta) = g_f(\theta) - g_h(\theta)$. As defined in the main text, $\delta(\theta)$ represents the difference in risk-adjusted wealth growth between the two countries. Given equilibrium asset pricing characterized by Eqs. (13) through (15), $\delta(\theta)$ is computed as.

$$\delta(\theta) = -\frac{(2\theta-1)\sigma^2\lambda^2\left(1-\lambda^2\right)\left(2-\lambda^2\right)}{2\Gamma(\theta)^2}. \tag{B.2}$$

We then introduce the following integrals:

$$I_1 = \int_{-\infty}^0 \exp\left[-\int_0^w c(u(y))dy\right]dw,$$
$$I_2 = \int_0^\infty \exp\left[-\int_0^w c(u(y))dy\right]dw,$$

and

$$M = \int_0^\infty \left[\frac{2}{G(u(w))^2}\exp\left[\int_0^w c(u(y))dy\right]\right]dw,$$

where.

$$c(u(y)) = \frac{2\delta(u(y))}{G(u(y))^2}, \tag{B.3}$$

and $u(y) = \frac{\exp(y)}{1+\exp(y)}$.

According to the above theorems of Skorokhod (1989), if $I_1 = \infty$, $I_2 = \infty$, and $M < \infty$, then κ has a unique ergodic distribution in $(-\infty, +\infty)$; accordingly, θ has a unique ergodic distribution in $(0,1)$.

A function $c(\ )$ characterized by equation (Appendix B.3) plays a key role in determining stationarity of κ or θ. Saito (1997) demonstrates that if $c(0) > 0$ and $c(1) < 0$, then κ (θ) has a unique ergodic distribution under some regulatory conditions. The process of κ or equation (Appendix B.1) always satisfies $c(0) > 0$ and $c(1) < 0$, because from equation (Appendix B.2), $\delta(0) = \frac{\sigma^2\lambda^2(2-\lambda^2)}{2(1-\lambda^2)} > 0$, and $\delta(1) = -\delta(0) < 0$, when $\lambda \neq 0$ and $1-\lambda^2 > 0$.

According to Gihman and Skorohod (1972), given the process of κ $(= \ln\frac{W_f}{W_h})$ or equation (Appendix B.1), a density function of κ is derived as

$$\frac{2\mu}{G(u(\kappa))^2}\exp\left[\int_0^\kappa c(u(y))dy\right],$$

where μ is chosen such that $\mu \int_0^\infty \left[\frac{2}{G(u(w))^2} \exp\left[\int_0^w c(u(y))dy\right]\right]dw = 1$. Figure B.1 depicts density functions of κ or $\ln \frac{W_f}{W_h}$ for $\lambda = 0.9, 0.8, 0.5$, and 0.3 with $\Delta_h = \Delta_f$ for the derivative trading model described in Section 2. According to this figure, the density function has a modal point at $\theta = 0.5$, but it has a fat tail on both ends.

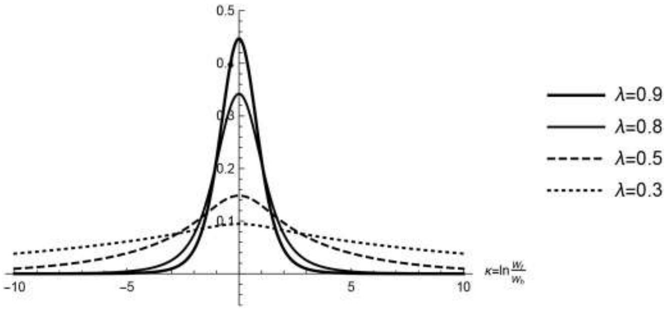

Figure B.1. Density Functions of κ $(= \ln \frac{W_f}{W_h+W_f})$ for Various Correlation Coefficients λ.

Appendix C. Relative Inflation Shocks and Real Exchange Rate Shocks as UIP Shocks

Nominal exchange rates $S(t)$ evolves according to

$$\frac{S(t+\Delta t)-S(t)}{S(t)} = \left[\frac{P_f(t+\Delta t)-P_f(t)}{P_f(t)} - \frac{P_h(t+\Delta t)-P_h(t)}{P_h(t)}\right] + \frac{\epsilon(t+\Delta t)-\epsilon(t)}{\epsilon(t)}.$$

Thus, $E_t\left[\frac{S(t+\Delta t)-S(t)}{S(t)}\right] = \Pi_f - \Pi_h$. The uncovered interest parity (UIP) shock $(U(t+\Delta t) - U(t))$ is defined as the disturbance on the UIP, and it is expressed as.

$$\begin{aligned} U(t+\Delta t)-U(t) &= \left\{R_h(t)\Delta t - R_f(t)\Delta t + \frac{S(t+\Delta t)-S(t)}{S(t)}\right\} - \left\{R_h(t)\Delta t - R_f(t)\Delta t + E_t\left[\frac{S(t+\Delta t)-S(t)}{S(t)}\right]\right\} \\ &= \left[v_f \frac{M_f(t+\Delta t)-M_f(t)}{M_f(t)} - v_h \frac{M_h(t+\Delta t)-M_h(t)}{M_h(t)}\right] + \xi\left[\frac{E_h(t+\Delta t)-E_h(t)}{E_h(t)} - \frac{E_f(t+\Delta t)-E_f(t)}{E_f(t)}\right]. \end{aligned}$$

As the above equation implies, the UIP shock consists of relative inflation shocks and real exchange rate shocks.

Given the arbitrage condition between domestic and foreign nominal bonds, $U(t+\Delta t)-U(t) + R_h(t)\Delta t - R_f(t)\Delta t + E_t\left[\frac{S(t+\Delta t)-S(t)}{S(t)}\right]$ is orthogonal to a stochastic discount factor, $\exp(-\rho\Delta t)\left[\frac{C_h(t+\Delta t)}{C_h(t)}\right]^{-1}$. That is,

$$E_t\left\{\left[U(t+\Delta t)-U(t) + R_h(t)\Delta t - R_f(t)\Delta t + E_t\left[\frac{S(t+\Delta t)-S(t)}{S(t)}\right]\right] \exp(-\rho\Delta t)\left[\frac{C_h(t+\Delta t)}{C_h(t)}\right]^{-1}\right\} = 0.$$

Accordingly, the following covered interest parity holds with the risk premium term $\Upsilon(t)\Delta t$.

$$R_h(t)\Delta t - R_f(t)\Delta t + E_t\left[\frac{S(t+\Delta t)-S(t)}{S(t)}\right] + \Upsilon(t)\Delta t = 0,$$

where $\Upsilon(t) = \frac{1}{E_t\left[\frac{C_h(t+\Delta t)}{C_h(t)}\right]^{-1}} E_t\left\{\left[\frac{C_h(t+\Delta t)}{C_h(t)}\right]^{-1}[U(t+\Delta t)-U(t)]\right\}$.

Appendix D. Nominal Bond Trading Equilibrium with Real Exchange Rate Shocks

To highlight the role of nominal bonds in fostering intertemporal trade, we will focus on the case where countries have identical drift and diffusion parameters, so that, $\alpha_h = \alpha_f = \alpha$, $\sigma_h = \sigma_f = \sigma$, $\Pi_h = \Pi_f = \Pi$, $v_h = v_f = v$, $\lambda_h = \lambda_f = \lambda$, $\phi_h = \phi_f = \phi$, and therefore $\Phi_h = \Phi_f = \Phi$.

With logarithmic utility, home country consumers follow the myopic consumption rule:

$$C = \rho W.$$

The optimal portfolio rules for the home country may be obtained as the solution to:

$$\begin{bmatrix} \omega_T^h \\ \omega_h^h \\ \omega_f^h \end{bmatrix} = \begin{bmatrix} \sigma^2 & \lambda\sigma v & \phi\sigma\xi \\ \lambda\sigma v & v^2 & 0 \\ \phi\sigma\xi & 0 & v^2+2\xi^2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha - r \\ R_h - \Pi - r \\ R_f - \Pi - r \end{bmatrix}. \tag{D.1}$$

A similar set of conditions hold for the foreign country.

$$\begin{bmatrix} \omega_T^f \\ \omega_h^f \\ \omega_f^f \end{bmatrix} = \begin{bmatrix} \sigma^2 & \phi\sigma\xi & \lambda\sigma v \\ \phi\sigma\xi & v^2+2\xi^2 & 0 \\ \lambda\sigma v & 0 & v^2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha - r \\ R_h - \Pi - r \\ R_f - \Pi - r \end{bmatrix} \tag{D.2}$$

Eqs. (7)–(9) and (D.1–D.2) are all linear, and can be solved as follows. For the equilibrium returns at the point of equal national wealth levels ($\theta = 0.5$),

$$\bar{r} = \alpha - \left[1 - \frac{(\lambda v - \phi\xi)^2}{2\left(v^2+\xi^2\right)}\right]\sigma^2, \tag{D.3}$$

$$\overline{R}_h = \overline{R}_f = \overline{R} = \bar{r} + \Pi + \frac{\lambda\sigma v\left(v^2+2\xi^2\right) + \phi\sigma v^2\xi}{2\left(v^2+\xi^2\right)}. \tag{D.4}$$

Note that $\bar{r} < \alpha$ because $2(v^2 + \xi^2) > (\phi\xi - \lambda v)^2$. Because the maximized risk-free rate is less than α, neither country takes short positions in its domestic technology.

In the case $0 < \theta < 1$,

$$R_h(\theta) = \overline{R} - \frac{\Phi}{2\left(v^2+\xi^2\right)\Psi(\theta)}\sigma v^2(2\theta-1)(\lambda v - \phi\xi)\left[v^2 + 2\theta\xi^2 - \sigma(2\theta-1)(\phi\xi - \lambda v)\right], \tag{D.5}$$

$$R_f(\theta) = \overline{R} + \frac{\Phi}{2\left(v^2+\xi^2\right)\Psi(\theta)}\sigma v^2(2\theta-1)(\lambda v - \phi\xi)\left[v^2 + 2(1-\theta)\xi^2 + \sigma(2\theta-1)(\phi\xi - \lambda v)\right], \tag{D.6}$$

$$r(\theta) = \bar{r} - \frac{\Phi}{2\left(v^2+\xi^2\right)\Psi(\theta)}\sigma^2 v^2(2\theta-1)^2(\lambda v - \phi\xi)^2, \tag{D.7}$$

where.

$$\Psi(\theta) = \theta(1-\theta)\left\{2v^2(\phi\xi - \lambda v)^2 + \xi^2\left[2(\phi\xi - \lambda v)^2 + 4\left(\lambda^2 v^2 + \left(1-\phi^2\right)\xi^2\right)\right]\right\} + v^2\Phi > 0. \tag{D.8}$$

Note that $\Psi(\theta)$ is always positive when $|\phi| < 1$, $0 < \theta < 1$, and $\Phi > 0$.

Using equations (Appendix D.5)-(Appendix D.7) and (Appendix D.1)-(Appendix D.2), we may derive the equilibrium portfolio holdings under nominal bond trade. The shares of wealth held in the home country technology, the home currency nominal bond, and the foreign currency nominal bond are written as follows. At the point of equal national wealth levels ($\theta = 0.5$),

$$\overline{\omega}_T^h = 1, \tag{D.9}$$

$$\overline{\omega}_h^h = -\overline{\omega}_f^h = -\frac{\sigma(\lambda v - \phi\xi)}{2\left(v^2+\xi^2\right)}. \tag{D.10}$$

Thus, the net positions in both nominal bonds ($\overline{\omega}_h^h + \overline{\omega}_f^h$) and real bonds ($1 - \overline{\omega}_T^h - \overline{\omega}_h^h - \overline{\omega}_f^h$) are zero at $\theta = 0.5$.

In the case $0 < \theta < 1$,

$$\omega_T^h(\theta) = 1 + \frac{1}{\Psi(\theta)}\theta(2\theta-1)v(\lambda v - \phi\xi)\left[\lambda\left(v^2+2\xi^2\right) - v\phi\xi\right], \tag{D.11}$$

$$\omega_h^h(\theta) = \frac{-1}{\Psi(\theta)}\theta\sigma(\lambda\nu-\phi\xi)\left[\left(1-(1-\theta)\lambda^2\right)\nu^2-\lambda\phi\nu\xi+\theta\xi^2\left(2-\phi^2\right)\right] \tag{D.12}$$

$$\omega_f^h(\theta) = \frac{1}{\Psi(\theta)}\theta\sigma(\lambda\nu-\phi\xi)\left[\left(1-\theta\lambda^2\right)\nu^2-\lambda\phi\nu\xi+(1-\theta)\xi^2\left(2-\phi^2\right)\right]. \tag{D.13}$$

Given equations (Appendix D.11)-(Appendix D.13), the net positions in nominal and real bonds are obtained as:

$$\omega_h^h(\theta)+\omega_f^h(\theta) = \frac{-1}{\Psi(\theta)}\theta(2\theta-1)\sigma(\lambda\nu-\phi\xi)\left(\lambda^2\nu^2+\xi^2\left(2-\phi^2\right)\right), \tag{D.14}$$

$$1-\omega_T^h-\omega_h^h(\theta)-\omega_f^h(\theta) = \frac{1}{\Psi(\theta)}\theta(2\theta-1)(\lambda\nu-\phi\xi)\left[-\lambda\left(\nu^3+2\nu\xi^2\right)+\lambda^2\sigma\nu^2+\phi\nu^2\xi+\sigma\xi^2\left(2-\phi^2\right)\right]. \tag{D.15}$$

In addition, a net foreign asset (NFA) position, $1-\omega_T^h$, is directly obtainable from equation (Appendix D.11).

We summarize the results of equations (D.5–D.7) and (D.11–D.15) in the following propositions:

Proof for Proposition 3

Proof. a) By direct inspection of equations (Appendix D.3) and (Appendix D.7), the real risk-free interest rate is minimized at $r^A = \alpha - \sigma^2$ when θ is 0 or 1, while it is maximized at $\bar{r} = \alpha - \left[1-\frac{(\phi\xi-\lambda\nu)^2}{2(\nu^2+\xi^2)}\right]\sigma^2$ when θ is 0.5. By $2(\nu^2+\xi^2) > (\phi\xi-\lambda\nu)^2$, $\bar{r}$ is below α. For $|\lambda| = 1$ and $\xi = 0$ ($|\phi| = 1$ and $\nu = 0$),[27] $\bar{r}$ reaches the complete markets real risk-free rate (r^C). However, if $\lambda = -\phi$, $\lambda^2 > 0.5$ to the extent that $\Phi > 0$, and $\nu = \xi$, then $\bar{r}$ exceeds r^C.

b) $A(\theta)$ is defined as $(1-(1-\theta)\lambda^2)\nu^2 - \lambda\phi\nu\xi + \theta\xi^2(2-\phi^2)$. $A'(\theta) > 0$, and $A(\theta)$ is minimized at $\theta = 0$. If $A(0) = (1-\lambda^2)\nu^2 - \lambda\phi\nu\xi > 0$, then $A(\theta) > 0$ for $0 < \theta < 1$. Similarly, $B(\theta) = (1-\theta\lambda^2)\nu^2 - \lambda\phi\nu\xi + (1-\theta)\xi^2(2-\phi^2)$ is decreasing in θ, and it is minimized at $\theta = 1$. If $B(1) = (1-\lambda^2)\nu^2 - \lambda\phi\nu\xi > 0$, then $B(\theta) > 0$ for $0 < \theta < 1$. Then, it follows from equations (Appendix D.12) and (Appendix D.13).

c) Follows directly from equation (Appendix D.14).

d) By equation (Appendix D.15), the sign of $1-\omega_T^h-\omega_h^h-\omega_f^h$ is indeterminate given θ. Taking for example the case $\phi\xi - \lambda\nu < 0$, and $\theta > 0.5$, if $\nu = 0$, then the sign is positive, but if $\xi = 0$, then it depends on the sign of $-\lambda\nu + \lambda^2\sigma$.

e) From equations (Appendix D.5) and (Appendix D.6), we have $\rho(\theta) = \frac{\Phi}{\Psi(\theta)}\sigma\nu^2(2\theta-1)(\phi\xi-\lambda\nu) > 0$ (<0) as $\theta < 0.5$ ($\theta > 0.5$).

f) Follows directly from equation (Appendix D.11).

When $\lambda\nu - \phi\xi < 0$ and other conditions unchanged, the opposite statements apply in parts b), c), e), and f) of the proposition. That is, the home country holds a long (short) position in home (foreign) currency bonds, and $\rho(\theta)$ is negative (positive) for $\theta < 0.5$ ($\theta > 0.5$).

Proof for Proposition 4

Proof. Let us apply Proposition Appendix B in Section 2 to this nominal bond trading equilibrium. Defining the difference between the foreign and home risk-adjusted growth rate as $\delta(\theta) = g_f(\theta) - g_h(\theta)$, we obtain $\delta(\theta)$ in this case.

$$\delta(\theta) = -\frac{\Phi}{2[\Psi(\theta)]^2}(2\theta-1)\nu^2\sigma^2(\lambda\nu-\phi\xi)^2\left[2\left(\nu^2+\xi^2\right)-(\phi\xi-\lambda\nu)^2\right]. \tag{D.16}$$

Since the denominator above and $2(\nu^2+\xi^2) - (\lambda\nu-\phi\xi)^2$ is always positive, therefore, $\Phi = (1-\lambda^2)(\nu^2+\xi^2) + (1-\lambda^2-\phi^2)\xi^2 > 0$ and $\lambda\nu \neq \phi\xi$ guarantee that $\delta(\theta)$ is positive (negative) for $\theta < 0.5$ ($\theta > 0.5$). Moreover, this satisfies the conditions

$$\delta(0) = \frac{1}{2\Phi\nu^2}\sigma^2(\phi\xi-\lambda\nu)^2\left[2\left(\nu^2+\xi^2\right)-(\lambda\nu-\phi\xi)^2\right] > 0,$$
$$\delta(1) = -\delta(0) < 0,$$

and $\delta(0.5) = 0$. Then, for $\lambda\nu \neq \phi\xi$, and $\Phi = (1-\lambda^2)(\nu^2+\xi^2) + (1-\lambda^2-\phi^2)\xi^2 > 0$, θ has a symmetric ergodic distribution in $(0,1)$ centered at $\theta = \frac{1}{2}$.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jinteco.2020.103386.

[27] If $\nu = 0$, then domestic currency bonds serve as risk-free bonds for domestic investors

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